

# Minimal intersection of curves on surfaces

Moira Chas \*

## Abstract

This paper is a consequence of the close connection between combinatorial group theory and the topology of surfaces.

In the eighties Goldman discovered a Lie algebra structure on the vector space generated by the free homotopy classes of oriented curves on an oriented surface. The Lie bracket  $[a, b]$  is defined as the signed sum over the intersection points of  $a$  and  $b$  of the loop product of  $a$  and  $b$  at the intersection points.

If one of the classes has a simple representative we give a combinatorial group theory description of the terms of the Lie bracket and prove that this bracket has as many terms, counted with multiplicity, as the minimal number of intersection points of  $a$  and  $b$ . In other words the bracket with a simple element has no cancellation and determines minimal intersection numbers. We show that analogous results hold for the Lie bracket (also discovered by Goldman) of unoriented curves. We give three applications: a factorization of Thurston's map defining the boundary of Teichmüller space, various decompositions of the underlying vector space of conjugacy classes into ad invariant subspaces and a connection between bijections of the set of conjugacy classes of curves on a surface preserving the Goldman bracket and the mapping class group.

2000 *Mathematics Subject Classification*. Primary: 57M99, 20E06.

*Key words and phrases*. Surfaces, simple closed curve, minimal intersection number, fundamental group, conjugacy classes, amalgamated free products, HNN extensions.

## 1 Introduction

Let  $a$  and  $b$  denote isotopy classes of embedded closed curves on a surface  $\Sigma$ . Denote by  $i(a, b)$  the minimal possible number of intersection points of curves representing  $a$  and  $b$ , where the intersections are counted with multiplicity. The function  $i(a, b)$  plays a central role in Thurston's work on low dimensional topology (see, for instance, [29], [12] and [14].) Let  $[a, b]$  denote the Lie bracket on the vector space of the free homotopy

---

\*This work is partially supported in part by NSF grant 1034525-1-29777.

classes of all essential directed closed curves on  $\Sigma$ . This Lie bracket originated from Wolpert's cosine formula, Thurston's earthquakes in Teichmüller space and Goldman's study of Poisson brackets (see [13].) The Lie algebra of oriented curves has been generalized using the loop product to a string bracket on the reduced  $\mathbb{S}^1$ -equivariant homology of the free loop space of any oriented manifold. (see [6] and [7].)

The main purpose of this paper is to prove that both, the Goldman Lie bracket on oriented curves and the Goldman Lie bracket on unoriented curves "count" the minimal number of intersection points of two simple curves.

In order to give a more precise statement of our results, let us review the definition of the Lie algebra for oriented curves discovered by Goldman: given two such free homotopy classes of directed curves on an orientable surface, take two representatives intersecting each other only in transverse double points. Each one of the intersection points will contribute a term to the geometric formula for the bracket. Each of these terms is defined as follows: take the conjugacy class of the curve obtained by multiplying the two curves at the intersection point and adjoin a negative sign if the orientation given by the ordered tangents at that point is different from the orientation of the surface. The bracket on  $W$ , the vector space of free homotopy classes is the bilinear extension of this construction. It is remarkable that this construction is well defined and satisfies skew-symmetry and Jacobi .

The bracket of unoriented curves can be defined on the subspace  $V$  of  $W$  fixed by the operation of reversing direction, as the restriction of the bracket. The subspace  $V$  is generated by elements of  $W$  of the form  $a + \bar{a}$  where  $a$  is a basis element and where  $\bar{a}$  denotes  $a$  with opposite direction. Let us identify unoriented curves up to homotopy with these expressions  $a + \bar{a}$ . The Lie bracket of two unoriented curves,  $a + \bar{a}$  and  $b + \bar{b}$ , is then defined geometrically by  $([a, b] + [\bar{a}, \bar{b}]) + ([a, \bar{b}] + [\bar{a}, b])$ , which equals  $[a + \bar{a}, b + \bar{b}]$  using  $[a, b] = [\bar{a}, \bar{b}]$ . An *unoriented term* is a term of the form  $c(z + \bar{z})$ , where  $c$  is an integer and  $z$  is a conjugacy class, that is, an element of the basis of  $V$  multiplied by an integer coefficient.

Since this Lie bracket uses the intersection points of curves, a natural problem was to study how well it reflects the intersection structure. In this regard, Goldman [13] proved the following result.

**Goldman's Theorem** *If the bracket of two free homotopy classes of curves (oriented or unoriented) is zero, and one of them has a simple representative, then the two classes have disjoint representatives.*

Goldman's proof uses the Kerckhoff earthquake convexity property of Teichmüller space [19] and in [13] he wondered whether this topological result had a topological proof. In [3] we gave such a proof when  $a$  was a non-separating simple closed curve on

a surface with non-empty boundary. Here we will give combinatorial proof of our Main Theorem (see below), which is a generalization of Goldman's result.

Now, suppose that  $a$  can be represented by a simple closed curve  $\alpha$ . Then for any free homotopy class  $b$  there exists a representative that can be written as a certain product which involves a sequence of elements in the fundamental group or groups of the connected components of  $\Sigma \setminus \alpha$  (see Sections 2 and 4 for precise definitions.) The number of terms of the sequence for  $b$  with respect to  $a$  is denoted by  $t(a, b)$ .

**Main Theorem** *Let  $a$  and  $b$  be two free homotopy classes of directed curves on an orientable surface. If  $a$  can be represented by a simple closed curve then the following nonnegative integers are equal:*

- (i) *The number of terms, counted with multiplicity, of the Goldman Lie bracket for oriented curves  $[a, b]$ .*
- (ii) *The number divided by two of unoriented terms (of the form  $x + \bar{x}$ ), counted with multiplicity, of the Goldman Lie bracket for unoriented curves  $[a + \bar{a}, b + \bar{b}]$ .*
- (iii) *The minimal number of intersection points of  $a$  and  $b$ ,  $i(a, b)$ .*
- (iv) *The number of terms of the sequence for  $b$  with respect to  $a$ ,  $t(a, b)$ .*

In particular, there is no cancellation of terms in the bracket of two curves, provided that one of them is simple.

As a consequence of our Main Theorem, we obtain the following result.

**Corollary of the Main Theorem** *If  $x$  and  $y$  are conjugacy classes of curves that can be represented by simple closed curves then  $t(x, y) = t(y, x)$ .*

We obtain the following global characterization of conjugacy classes containing simple representatives.

**Corollary of the Main Theorem** *Let  $a$  denote a free homotopy class of curves on a surface. Then the following statements are equivalent.*

- (1) *The class  $a$  has a representative that is a power of a simple curve.*
- (2) *For every free homotopy class  $b$  the (oriented) bracket  $[a, b]$  has many as oriented terms counted with multiplicity as the minimal intersection number of  $a$  and  $b$ .*
- (3) *For every free homotopy class  $b$  the (unoriented) bracket  $[a, b]$  has many as unoriented terms counted with multiplicity as twice the minimal intersection number of  $a$  and  $b$ .*

In [5] a local characterization of simple closed curves in terms of the Lie bracket will be given. Actually, the problem of characterizing algebraically embedded conjugacy classes was the original motivation of [3], [6] and [7].

Here is a brief outline of the arguments we follow to prove the main theorem.

- (1) The key point is that when a curve is simple, we can apply either HNN extensions or amalgamated products to write elements of the fundamental group of the surface as a product that involves certain sequences of elements of subgroups which are the fundamental group of the connected components of the surface minus the simple curve.
- (2) Using combinatorial group theory tools we show that if certain equations do not hold in an HNN extension or an amalgamated free product then certain products of the sequences in (1) cannot be conjugate, (Sections 2 and 4.)
- (3) We show that each of the terms of the bracket can be obtained by inserting the conjugacy class of the simple curve in different places of the sequences in (1) made circular. (Sections 3 and 5.)
- (4) We show that the equations of (2) do not hold in the HNN extensions and amalgamated free products determined by a simple closed curve. (Section 6)

The Goldman bracket extends to higher dimensional manifolds as one of the String Topology operations. Abbaspour [1] characterizes hyperbolic three manifolds among closed three manifolds using the loop product which is another String Topology operation. Some of his arguments rely on the decomposition of the fundamental group of a manifold into amalgamated products based on torus submanifolds, and the use of this decomposition to give expressions for certain elements of the fundamental group, which in term, gives a way of computing the loop product.

Here is the organization of this work: in Section 2 we list the known results about amalgamated products of groups we will use, and we prove that certain equations do not hold in such groups. In Section 3, we apply the results of the previous section to find a combinatorial description of the Goldman bracket of two oriented curves, one of them simple and separating (see Figure 3.) In Section 4 we list results concerning HNN extensions and we prove that certain equations do not hold in such groups. In Section 5, we apply the results of Section 4 to describe combinatorially the bracket of an oriented non-separating simple closed curve with another oriented curve (see Figure 8.) In Section 6 we prove propositions about the fundamental group of the surface which will be used to show that our sequences satisfy the hypothesis of the main theorems of 2 and 4. In Section 7 we put together most of the above results to show that there is no cancellation in the Goldman bracket of two directed curves, provided that one of them is simple. In Section 8 we define the bracket of unoriented curves and prove

that there is no cancellation if one of the curves is simple. In Section 9 we exhibit some examples that show that the hypothesis of one of the curves being simple cannot be dropped (see Figure 9.) In the next three sections, we exhibit some applications of our main results. More precisely, in Section 10, we show how our main theorem yields a factorization of Thurston's map on the set of all simple conjugacy classes of curves on a surface, through the power of the vector space of all conjugacy classes to the set of simple conjugacy classes. In Section 11 we exhibit several partitions of the vector space generated by all conjugacy classes, which are invariant under certain Lie algebra operations. We conclude by showing in Section 12 that if a function on the set of conjugacy classes preserves the bracket, then is determined by an element of the mapping class group of the surface. We conclude by stating some problems and open questions in Section 13.

*Acknowledgments:* This work benefitted from stimulating exchanges with Pavel Etingof and Dennis Sullivan and especially Warren Dicks who suggested an improved demonstration of Theorem 2.13.

## 2 Amalgamated products

This section deals exclusively with results concerning Combinatorial Group Theory. We start by stating definitions and known results about amalgamating free products. Using these tools, we prove the main results of this section, namely, Theorems 2.13 and 2.15. These two theorems state that certain pairs of elements of an amalgamating free product are not conjugate. By Theorem 3.4 if  $b$  is an arbitrary conjugacy class and  $a$  is a conjugacy class containing a separating simple representative, then the Goldman Lie bracket  $[a, b]$  can be written as an algebraic sum  $t_1 + t_2 + \cdots + t_n$ , with the following property: If there exist two terms  $t_i$  and  $t_j$  that cancel, then the conjugacy classes associated to the terms  $t_i$  and  $t_j$  both satisfy the hypothesis of Theorem 2.13 or both satisfy the hypothesis of Theorem 2.15. This will show that the terms of the Goldman Lie bracket exhibited in Theorem 3.4 are all distinct.

Alternatively, one could make use of the theory of groups acting on graphs (see, for instance, [10]) to prove Theorems 2.13 and 2.15.

Let  $C, G$  and  $H$  be groups and let  $\varphi: C \longrightarrow G$  and  $\psi: C \longrightarrow H$  be monomorphisms. We denote the *free product of  $G$  and  $H$  amalgamating the subgroup  $C$  (and morphisms  $\varphi$  and  $\psi$ )* by  $G *_C H$ . This group is defined as the quotient of the free product  $G * H$  by the normal subgroup generated by  $\varphi(c)\psi(c)^{-1}$  for all  $c \in C$ . (See [16],[25], [26] or [8] for detailed definitions.)

Since there are canonical injective maps from  $C, G$  and  $H$  to  $G *_C H$ , in order to

make the notation lighter we will work as if  $C$ ,  $G$  and  $H$  were included in  $G *_C H$ .

**Definition 2.1.** Let  $n$  be a non-negative integer. A finite sequence  $(w_1, w_2, \dots, w_n)$  of elements of  $G *_C H$  is *reduced* if the following conditions hold,

- (1) Each  $w_i$  is in one of the factors,  $G$  or  $H$ .
- (2) For each  $i \in \{1, 2, \dots, n-1\}$ ,  $w_i$  and  $w_{i+1}$  are not in the same factor.
- (3) If  $n = 1$ , then  $w_1$  is not the identity.

□

The case  $n = 0$  is included as the empty sequence. Also, if  $n$  is larger than one then for each  $i \in \{1, 2, \dots, n\}$ ,  $w_i \notin C$ , other wise, (2) is violated.

The proof of the next theorem can be found in [25], [8] or [26].

**Theorem 2.2.** (1) Every element of  $G *_C H$  can be written as a product  $w_1 w_2 \cdots w_n$  where  $(w_1, w_2, \dots, w_n)$  is a reduced sequence.  
(2) If  $n$  is a positive integer and  $(w_1, w_2, \dots, w_n)$  is a reduced sequence then the product  $w_1 w_2 \cdots w_n$  is not the identity.

We could not find a proof of the next well known result in the literature, so we include it here.

**Theorem 2.3.** If  $(w_1, w_2, \dots, w_n)$  and  $(h_1, h_2, \dots, h_n)$  are reduced sequences such the products  $w_1 w_2 \cdots w_n$  and  $h_1 h_2 \cdots h_n$  are equal then exists a finite sequence of element of  $C$ , namely  $(c_1, c_2, \dots, c_{n-1})$  such that

- (i)  $w_1 = h_1 c_1$ ,
- (ii)  $w_n = c_{n-1}^{-1} h_n$ ,
- (iii) For each  $i \in \{2, 3, \dots, n-1\}$ ,  $w_i = c_{i-1}^{-1} h_i c_i$ .

*Proof.* We can assume  $n > 1$ . Since the products are equal,  $h_n^{-1} h_{n-1}^{-1} \cdots h_1^{-1} w_1 w_2 \cdots w_n$  is the identity. By Theorem 2.2(2), the sequence

$$(h_n^{-1}, h_{n-1}^{-1}, \dots, h_1^{-1}, w_1, w_2, \dots, w_n)$$

is not reduced. Since the sequences  $(h_n^{-1}, h_{n-1}^{-1}, \dots, h_1^{-1})$  and  $(w_1, w_2, \dots, w_n)$  are reduced,  $w_1$  and  $h_1^{-1}$  belong both to  $G \setminus C$  or both to  $H \setminus C$ . Assume the first possibility holds, that is  $h_1$  and  $w_1$  are in  $G \setminus C$ , (the second possibility is treated analogously.) Set  $c_1 = h_1^{-1} w_1$ . By our assumption,  $c_1 \in G$ . The sequence

$$(h_n^{-1}, h_{n-1}^{-1}, \dots, h_2^{-1}, c_1, w_2, \dots, w_n)$$

has product equal to the identity. All the elements of this sequence are alternatively in  $G \setminus C$  and  $H \setminus C$ , with the possible exception of  $c_1$ . By Theorem 2.2(2), this sequence is not reduced. Then  $c_1 \notin G \setminus C$ . Since  $c_1 \in G$ ,  $c_1 \in C$ .

Now, consider the sequence

$$(h_n^{-1}, h_{n-1}^{-1}, \dots, h_3^{-1}, h_2^{-1} c_1 w_2, w_3, \dots, w_n)$$

By arguments analogous to those we made before,  $h_2^{-1} c_1 w_2 \in C$ . Denote  $c_2 = h_2^{-1} c_1 w_2$ . Thus,  $w_2 = c_1^{-1} h_2 c_2$

This shows that we can apply induction to find the sequence  $(c_1, c_2, \dots, c_{n-1})$  claimed in the theorem. ■

**Definition 2.4.** A finite sequence of elements  $(w_1, w_2, \dots, w_n)$  of  $G *_C H$  is *cyclically reduced* if every cyclic permutation of  $(w_1, w_2, \dots, w_n)$  is reduced. □

**Notation 2.5.** When dealing with free products with amalgamation, every time we consider a sequence of the form  $(a_1, a_2, \dots, a_n)$  we take subindexes mod  $n$  in the following way: For each  $j \in \mathbb{Z}$ , by  $a_j$  we will denote  $a_i$  where  $i$  is the only integer in  $\{1, 2, \dots, n\}$  such that  $n$  divides  $i - j$ . □

**Remark 2.6.** If  $(w_1, w_2, \dots, w_n)$  is a cyclically reduced sequence and  $n \neq 1$  then  $n$  is even. Also, for every pair of integers  $i$  and  $j$ ,  $w_i$  and  $w_j$  are both in  $G$  or both in  $H$  if and only if  $i$  and  $j$  have the same parity. □

The following result is a direct consequence of Theorem 2.2(1) and [26, Theorem 4.6].

**Theorem 2.7.** *Let  $s$  be a conjugacy class of  $G *_C H$ . Then there exists a cyclically reduced sequence such that the product is a representative of  $s$ . Moreover every cyclically reduced sequence with product in  $s$  has the same number of terms.*

The following result gives necessary conditions for two cyclically reduced sequences to be conjugate.

**Theorem 2.8.** *Let  $n \geq 2$  and let  $(w_1, w_2, \dots, w_n)$  and  $(v_1, v_2, \dots, v_n)$  be cyclically reduced sequences such that the products  $w_1 w_2 \cdots w_n$  and  $v_1 v_2 \cdots v_n$  are conjugate. Then there exists an integer  $k \in \{0, 1, \dots, n-1\}$  and a sequence of elements of  $C$ ,  $c_1, c_2, \dots, c_n$  such that for each  $i \in \{1, 2, \dots, n\}$ ,  $w_i = c_{i+k-1}^{-1} v_{i+k} c_{i+k}$ . In particular, for each  $i \in \{1, 2, \dots, n\}$ ,  $w_i$  and  $v_{i+k}$  are both in  $G$  or both in  $H$ .*

*Proof.* By [25, Theorem 2.8], there exist  $k \in \{0, 1, 2, \dots, n-1\}$  and an element  $c$  in the amalgamating group  $C$  such that

$$w_1 w_2 \cdots w_n = c^{-1} v_{k+1} v_{k+2} \cdots v_{k+n-1} v_{k+n} c.$$

The sequences  $(w_1, w_2, \dots, w_n)$  and  $(c^{-1} v_{k+1}, v_{k+2}, \dots, v_{k+n-1}, v_{k+n} c)$  are reduced and have the same product. By Theorem 2.3 there exists a sequence of elements of  $C$ ,  $(c_1, c_2, \dots, c_{n-1})$  such that

- (i)  $w_1 = c^{-1} v_{k+1} c_1$ ,
- (ii)  $w_n = c_{n-1}^{-1} v_{k+n} c$ ,
- (iii) For each  $i \in \{2, 3, \dots, n-1\}$ ,  $w_i = c_{i-1}^{-1} v_{k+i} c_i$ .

Relabeling the sequence  $(c, c_1, c_2, \dots, c_{n-1})$  we obtain the desired result.  $\blacksquare$

If  $C_1$  and  $C_2$  are subgroups of a group  $G$  a *double coset of  $G$  mod  $C_1$  on the left and  $C_2$  on the right* or briefly a *double coset of  $G$*  is an equivalence class of the equivalence relation on  $G$  defined for each pair of elements  $x$  and  $y$  of  $G$  by  $x \sim y$  if there exist  $c_1 \in C_1$  and  $c_2 \in C_2$  such that  $x = c_1 y c_2$ . If  $x \in G$ , then the equivalence class containing  $x$  is denoted by  $C_1 x C_2$ . Using this notation, the next corollary follows directly from Theorem 2.8.

**Corollary 2.9.** *Let  $n \geq 2$  and let  $(w_1, w_2, \dots, w_n)$  and  $(v_1, v_2, \dots, v_n)$  be cyclically reduced sequences such that the products  $w_1 w_2 \cdots w_n$  and  $v_1 v_2 \cdots v_n$  are conjugate. Then*

$$\{C w_1 w_2 C, C w_2 w_3 C, \dots, C w_n w_1 C\} = \{C v_1 v_2 C, C v_2 v_3 C, \dots, C v_n v_1 C\}.$$

**Remark 2.10.** Observe that a result stronger than Corollary 2.9 holds, namely, one can associate a unique cyclic sequence of double cosets (and not only a set) to each conjugacy class.  $\square$

**Definition 2.11.** Let  $C$  be a subgroup of a group  $G$  and let  $g$  be an element of  $G$ . Denote by  $C^g$  the subgroup of  $G$  defined by  $g^{-1} C g$ . We say that  $C$  is *malnormal* in  $G$  if  $C^g \cap C = \{1\}$  for every  $g \in G \setminus C$ .  $\square$

**Lemma 2.12.** *Let  $G *_C H$  be a free product with amalgamation such that the amalgamating group  $C$  is malnormal in  $G$  and is malnormal in  $H$ . Let  $a$  and  $b$  be elements of  $C$ . Let  $w_1, w_2$  and  $v_1, v_2$  be two reduced sequences such that the sets of double cosets*

$$\{C w_1 a w_2 C, C v_1 v_2 C\} \text{ and } \{C w_1 w_2 C, C v_1 b v_2 C\}$$

*are equal. Then  $a$  and  $b$  are conjugate in  $C$ . Moreover, if  $a \neq 1$  or  $b \neq 1$  then  $v_1$  and  $w_1$  are both in  $G$  or both in  $H$ .*



*Proof.* We claim that if  $Cw_1aw_2C = Cw_1w_2C$  then  $a = 1$ . Indeed, if  $Cw_1aw_2C = Cw_1w_2C$  then there exist  $c_1$  and  $c_2$  in  $C$  such that  $c_1w_1aw_2c_2 = w_1w_2$ . By Theorem 2.3 there exists  $d \in C$  such that

$$w_1 = c_1w_1ad \quad (2.1)$$

$$w_2 = d^{-1}w_2c_2 \quad (2.2)$$

Since  $C$  is malnormal in  $H$ , by Equation (2.2),  $d = 1$ . Thus Equation (2.1) becomes  $w_1 = c_1w_1a$ . By malnormality,  $a = 1$  and the proof of the claim is complete.

Assume first that  $Cw_1aw_2C = Cw_1w_2C$  and  $Cv_1v_2C = Cv_1bv_2C$  then by the claim,  $a = 1$  and  $b = 1$ . Thus,  $a$  and  $b$  are conjugate in  $C$  and the result holds in this case.

Now, assume that  $Cw_1aw_2C = Cv_1bv_2C$  and  $Cv_1v_2C = Cw_1w_2C$ . By definition of double cosets, there exist  $c_1, c_2, c_3, c_4 \in C$  such that  $v_1bv_2 = c_1w_1aw_2c_2$  and  $v_1v_2 = c_3w_1w_2c_4$ . By Theorem 2.3, there exist  $d_1, d_2 \in C$  such that

$$v_1b = c_1w_1ad_1 \quad (2.3)$$

$$v_2 = d_1^{-1}w_2c_2 \quad (2.4)$$

$$v_1 = c_3w_1d_2 \quad (2.5)$$

$$v_2 = d_2^{-1}w_2c_4 \quad (2.6)$$

By Equations (2.4) and (2.6) and malnormality,  $d_1 = d_2$ . By Equations (2.3) and (2.5) and malnormality,  $d_1 = ad_1b^{-1}$ . Thus,  $a$  and  $b$  are conjugate in  $C$ . Finally by Equation (2.3), since  $C$  is a subgroup of  $G$  and a subgroup of  $H$ ,  $w_1$  and  $v_1$  are both in  $G$  or both in  $H$ . ■

Now we will show that certain pairs of conjugacy classes are distinct by proving that the associated sequences of double cosets of some of their respective representatives are distinct. Warren Dicks suggested the idea of this proof. Our initial proof [4] applied repeatedly the equations given by Theorem 2.8 to derive a contradiction.

**Theorem 2.13.** *Let  $G *_C H$  be a free product with amalgamation. Let  $i$  and  $j$  be distinct elements of  $\{1, 2, \dots, n\}$  and let  $a$  and  $b$  be elements of  $C$ . Assume all the following:*

- (1) *The subgroup  $C$  is malnormal in  $G$  and is malnormal in  $H$ .*
- (2) *If  $i$  and  $j$  have the same parity then  $a$  and  $b$  are not conjugate in  $C$ .*
- (3) *Either  $a \neq 1$  or  $b \neq 1$ .*

Then for every cyclically reduced sequence  $(w_1, w_2, \dots, w_n)$  the products

$$w_1 w_2 \cdots w_i a w_{i+1} \cdots w_n \text{ and } w_1 w_2 \cdots w_j b w_{j+1} \cdots w_n$$

are not conjugate.

*Proof.* Let  $(w_1, w_2, \dots, w_n)$  be a cyclically reduced sequence. Assume that there exist  $a$  and  $b$  in  $C$  and  $i, j \in \{1, 2, \dots, n\}$  as in the hypothesis of the theorem such that the products  $w_1 w_2 \cdots w_i a w_{i+1} \cdots w_n$  and  $w_1 w_2 \cdots w_j b w_{j+1} \cdots w_n$  are conjugate.

By Corollary 2.9, the sequences of double cosets mod  $C$  on the right and the left associated with  $(w_1, w_2, \dots, a w_{i+1}, \dots, w_n)$  and with  $(w_1, w_2, \dots, b w_{j+1}, \dots, w_n)$  are equal. In symbols,

$$\{C w_1 w_2 C, \dots, C w_i a w_{i+1} C, \dots, C w_n w_1 C\} = \{C w_1 w_2 C, \dots, C w_j b w_{j+1} C, \dots, C w_n w_1 C\}$$

Removing from both sets the elements that are denoted by equal expressions we obtain

$$\{C w_i a w_{i+1} C, C w_j w_{j+1} C\} = \{C w_j b w_{j+1} C, C w_i w_{i+1} C\}.$$

By Lemma 2.12,  $a$  and  $b$  are conjugate in  $C$ . Moreover,  $w_i$  and  $w_j$  are both in  $G$  or both in  $H$ . By Remark 2.6,  $i$  and  $j$  have the same parity, contradicting our hypothesis (2). ■

**Remark 2.14.** It is not hard to construct an example that shows that hypothesis (1) of Theorem 2.13 is necessary. Indeed take, for instance,  $G$  and  $H$  two infinite cyclic groups generated by  $x$  and  $y$  respectively. Let  $C$  be an infinite cyclic subgroup generated by  $z$ . Define  $\varphi: C \rightarrow G$  and  $\psi: C \rightarrow H$  by  $\varphi(z) = x^2$  and  $\psi(z) = y^3$ . The sequence  $(x, y)$  is cyclically reduced. Let  $a = x^2$  and  $b = x^2$ . The products  $xay$  and  $xyb$  are conjugate.

The following example shows that hypothesis (2) of Theorem 2.13 is necessary. Let  $G *_C H$  be a free product with amalgamation, let  $c$  be an element of  $C$  and let  $(w_1, w_2)$  be a reduced sequence. Thus,  $(w_1, w_2, w_1, w_2)$  is a cyclically reduced sequence. On the other hand, the products  $w_1 c w_2 w_1 w_2$  and  $w_1 w_2 w_1 c w_2$  are conjugate. □

The next result states certain elements of an amalgamated free product are not conjugate.

**Theorem 2.15.** *Let  $G *_C H$  be a free product with amalgamation. Let  $a$  and  $b$  be elements of  $C$  and let  $i, j \in \{1, 2, \dots, n\}$ . Assume that for every  $g \in (G \cup H) \setminus C$ ,  $a$  and  $b$  are not in the same double coset relative to  $C$ , i.e.,  $g^{-1} a g \notin C g C$ . Then for every cyclically reduced sequence  $(w_1, w_2, \dots, w_n)$  the products*

$$w_1 w_2 \cdots w_i a w_{i+1} \cdots w_n \text{ and } w_n^{-1} w_{n-1}^{-1} \cdots w_{j+1}^{-1} b w_j^{-1} \cdots w_1^{-1}$$

are not conjugate.

*Proof.* Assume that the two products are conjugate. Observe that the sequences

$$(w_1, w_2, \dots, w_i a, w_{i+1}, \dots, w_n) \text{ and } (w_n^{-1}, w_{n-1}^{-1}, \dots, w_{j+1}^{-1} b, w_j^{-1}, \dots, w_1^{-1})$$

are cyclically reduced. By Theorem 2.8 there exists an integer  $k$  such that for every  $h \in \{1, 2, \dots, n\}$   $w_h$  and  $w_{1-h+k}^{-1}$  are both in  $G$  or both in  $H$ . Then  $w_h$  and  $w_{1-h+k}$  are both in  $G$  or both in  $H$ . By Remark 2.6,  $n$  is even and  $h$  and  $1 - h + k$  have the same parity. Thus,  $k$  is odd.

Set  $l = \frac{k+1}{2}$ . By Theorem 2.8,  $w_l$  and  $w_{1-l+k}^{-1} = w_l^{-1}$  are in the same double coset of  $G$  (if  $l \in \{i, j-k\}$ , either  $w_l$  or  $w_{1-l+k}$  may appear multiplied by  $a$  or  $b$  in the equations of Theorem 2.8 but this does not change the double coset.) Thus  $w_l$  and  $w_l^{-1}$  are in the same double coset mod  $C$  of  $G$ , contradicting our hypothesis. ■

### 3 Oriented separating simple loops

The goal of this section is to prove Theorem 3.4, which gives us a way to compute the bracket of a separating simple closed curve  $x$  and the product of the terms of a cyclically reduced sequence given by the amalgamated free product that  $x$  determines. We prove this theorem by finding in Lemma 3.2 appropriate representatives for the separating simple closed curve  $x$  and the terms of the cyclically reduced sequence.

Through the rest of these pages, a *surface* will mean a connected oriented surface. We denote such a surface by  $\Sigma$ . The fundamental group of  $\Sigma$  will be denoted by  $\pi_1(\Sigma, p)$  where  $p \in \Sigma$  is the basepoint. By a *curve* we will mean a closed oriented curve on  $\Sigma$ . We will use the same letter to denote a curve and its image on  $\Sigma$ .

Let  $\chi$  be a separating non-trivial simple curve on  $\Sigma$ , non-parallel to a boundary component of  $\Sigma$ . Choose a point  $p \in \chi$  to be the basepoint of each of the fundamental groups which will appear in this context. Denote by  $\Sigma_1$  the union of  $\chi$  and one of the connected components of  $\Sigma \setminus \chi$  and by  $\Sigma_2$  the union of  $\chi$  with the other connected component.

**Remark 3.1.** As a consequence of the van Kampen's theorem (see [16])  $\pi_1(\Sigma, p)$  is canonically isomorphic to the free product of  $\pi_1(\Sigma_1, p)$  and  $\pi_1(\Sigma_2, p)$  amalgamating the subgroup  $\pi_1(\chi, p)$ , where the monomorphisms  $\pi_1(\chi, p) \longrightarrow \pi_1(\Sigma_1, p)$  and  $\pi_1(\chi, p) \longrightarrow \pi_1(\Sigma_2, p)$  are the induced by the respective inclusions. □

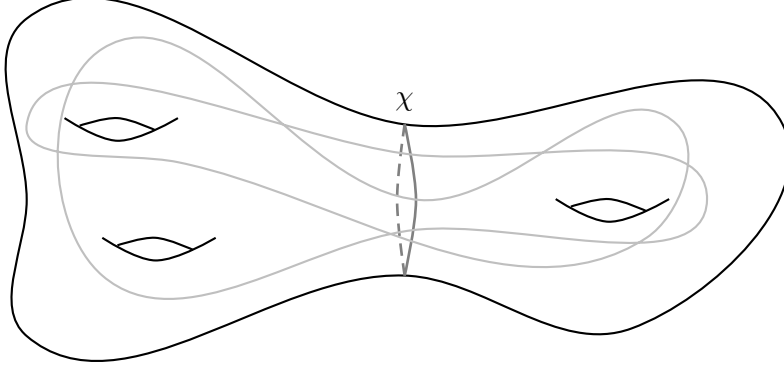


Figure 1: A separating curve  $\chi$  intersecting another curve

**Lemma 3.2.** *Let  $\chi$  be a separating simple closed curve on  $\Sigma$ . Let  $(w_1, w_2, \dots, w_n)$  be a cyclically reduced sequence for the amalgamated product of Remark 3.1. Then there exists a sequence of curves  $(\gamma_1, \gamma_2, \dots, \gamma_n)$  such that for each  $i \in \{1, 2, \dots, n\}$  the all the following holds.*

- (1) *The curve  $\gamma_i$  is a representative of  $w_i$ .*
- (2) *The curve  $\gamma_i$  is alternately in  $\Sigma_1$  and  $\Sigma_2$ .*
- (3) *The point  $p$  is the basepoint of  $\gamma_i$ .*
- (4) *The point  $p$  is not a self-intersection point of  $\gamma_i$ . In other words,  $\gamma_i$  passes through  $p$  exactly once.*

*Moreover, the product  $\gamma_1\gamma_2\cdots\gamma_n$  is a representative of the product  $w_1w_2\cdots w_n$  and the curve  $\gamma_1\gamma_2\cdots\gamma_n$  and  $w_1w_2\cdots w_n$  intersect  $\chi$  transversally with multiplicity  $n$ .*

*Proof.* For each  $i \in \{1, 2, \dots, n\}$ , take a representative  $\gamma_i$  in of  $w_i$ . Notice that  $\gamma_i \subset \Sigma_1$  or  $\gamma_i \subset \Sigma_2$ . We homotope  $\gamma_i$  if necessary, in such a way that  $\gamma_i$  passes through  $p$  only once, at the basepoint  $p$

Since each  $\gamma_i$  intersects  $\chi$  only at  $p$ , the product  $\gamma_1\gamma_2\cdots\gamma_n$  intersects  $\chi$  exactly  $n$  times. Each of this intersections happens when the curve  $\gamma_1\gamma_2\cdots\gamma_n$  passes from  $\Sigma_1$  to  $\Sigma_2$  or from  $\Sigma_2$  to  $\Sigma_1$ . This implies that these  $n$  intersection points of  $\chi$  with  $\gamma_1\gamma_2\cdots\gamma_n$  are transversal. ■

Let  $\Sigma$  be an oriented surface. The Goldman bracket [13] is a Lie bracket defined on the vector space generated by all free homotopy classes of oriented curves on the

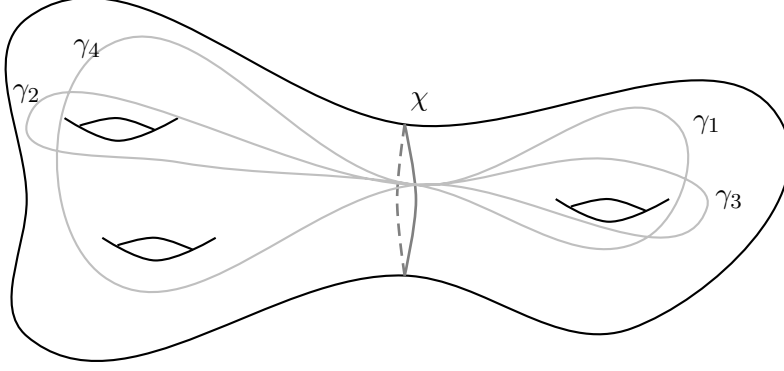


Figure 2: The representative of Lemma 3.2

surface  $\Sigma$ . We recall the definition: For each pair of homotopy classes  $a$  and  $b$ , consider representatives  $a$  and  $b$  respectively, that only intersect in transversal double points. The bracket of  $[a, b]$  is defined as the signed sum over all intersection points  $P$  of  $a$  and  $b$  of free homotopy class of the curve that goes around  $a$  starting and ending at  $P$  and then goes around  $b$  starting and ending at  $P$ . The sign of the term at an intersection point  $P$  is the intersection number of  $a$  and  $b$  at  $P$ . (See Figure 3.)

The above definition can be extended to consider pairs of representatives where branches intersect transversally but triple (and higher) points are allowed. Indeed, take a pair of such representatives  $a$  and  $b$ . The Goldman Lie bracket is the sum over the intersection of pairs of small arcs, of the conjugacy classes of the curve obtained by starting in an intersection point and going along the  $a$  starting in the direction of the first arc, and then going around  $b$  starting in the direction of the second arc. The sign is determined by the pair of tangents of the ordered oriented arcs at the intersection point.

**Notation 3.3.** The Goldman bracket  $[a, b]$  is computed for pairs of conjugacy classes  $a$  and  $b$  of curves on  $\Sigma$ . In order to make the notation lighter, we will abuse notation by writing  $[u, v]$ , where  $u, v$  are elements of the fundamental group of  $\Sigma$ . By  $[u, v]$ , then, we will mean the bracket of the conjugacy class of  $u$  and the conjugacy class of  $v$ .  $\square$

**Theorem 3.4.** *Let  $x$  be a conjugacy class of  $\pi_1(\Sigma, p)$  which can be represented by a separating simple closed curve  $\chi$ . Let  $(w_1, w_2, \dots, w_n)$  be a cyclically reduced sequence of the for the amalgamated product determined by  $\chi$  in Remark 3.1. Then  $[w_1, x] = 0$ .*

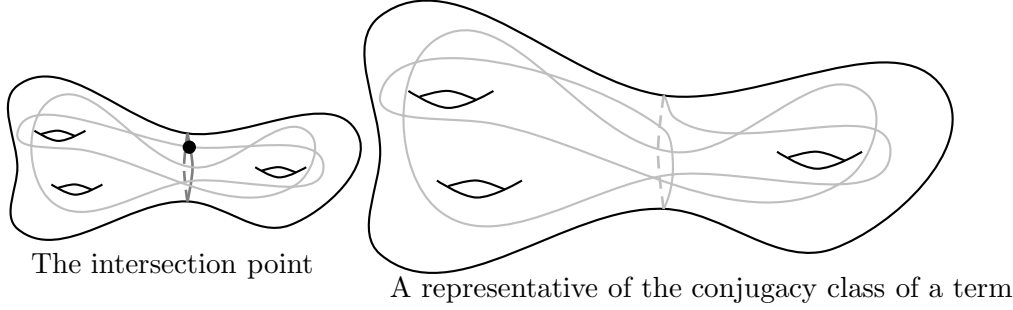


Figure 3: An intersection point (left) and the corresponding term of the bracket (right)

Moreover, if  $n > 1$  then there exists  $s \in \{1, -1\}$  such that the bracket is given by

$$s[w_1 w_2 \cdots w_n, x] = \sum_{i=1}^n (-1)^i w_1 w_2 \cdots w_i x w_{i+1} \cdots w_n.$$

*Proof.* For each curve  $\gamma$  there exists a representative of the null-homotopic class disjoint from  $\gamma$  so the result holds when  $n = 0$ . We prove now that  $[w_1, x] = 0$ . By definition of a cyclically reduced sequence,  $w_1 \in \pi_1(\Sigma_1, p)$  or  $w_1 \in \pi_1(\Sigma_2, p)$ . Suppose that  $w_1 \in \pi_1(\Sigma_1, p)$  (the other possibility is analogous.) Choose a curve  $\gamma_1 \subset \Sigma_1$  which is a representative of  $w_1$ . Since  $\gamma_1 \subset \Sigma_1$ , we can homotope  $\gamma_1$  to a curve which has no intersection with  $\chi$ , (clearly, this is a free homotopy which does not fix the basepoint  $p$ .) This shows that  $w_1$  and  $x$  have disjoint representatives, and then  $[w_1, x] = 0$ .

Now, assume that  $n > 1$ . Let  $(\gamma_1, \gamma_2, \dots, \gamma_n)$  be the sequence given by Lemma 3.2 for  $(w_1, w_2, \dots, w_n)$ .

The loop product  $\gamma_1 \gamma_2 \cdots \gamma_n$  is a representative of the group product  $w_1 w_2 \cdots w_n$ .

Every intersection of  $\gamma_1 \gamma_2 \cdots \gamma_n$  with  $\chi$  occurs when  $\gamma_1 \gamma_2 \cdots \gamma_n$  leaves one connected component of  $\Sigma \setminus \chi$  to enter the other connected component. That is, between each  $\gamma_i$  and  $\gamma_{i+1}$ . (Recall we are using Notation 2.5 so the intersection between  $\gamma_n$  and  $\gamma_1$  is considered.)

For each  $i \in \{1, 2, \dots, n\}$  denote by  $p_i$  the intersection point of  $\chi$  and  $\gamma_1 \gamma_2 \cdots \gamma_n$  between  $\gamma_i$  and  $\gamma_{i+1}$ . The loop product  $\gamma_1 \gamma_2 \cdots \gamma_i \chi \gamma_{i+1} \cdots \gamma_n$  is a representative of the conjugacy class of the term of the Goldman Lie bracket corresponding to  $p_i$ . Thus, for each  $i \in \{1, 2, \dots, n\}$ , the conjugacy class of the term of the bracket corresponding to the intersection point  $p_i$  has  $w_1 \cdots w_i x w_{i+1} \cdots w_n$  as representative.

Let  $i, j \in \{1, 2, \dots, n\}$  with different parity. Assume that  $w_i \in \pi_1(\Sigma_1, p)$ , (the case  $w_i \in \pi_2(\Sigma_1, p)$  is similar.) The tangent vector of  $\gamma_1 \gamma_2 \cdots \gamma_n$  at  $p_i$  points towards  $\Sigma_2$  and

the tangent vector of  $\gamma_1\gamma_2\cdots\gamma_n$  at  $p_j$  points towards  $\Sigma_1$ . This shows that the signs of the bracket terms corresponding to  $p_i$  and  $p_j$  are opposite, completing the proof. ■

**Remark 3.5.** In Theorem 3.4, all the intersections of the chosen representatives of  $w_1w_2\cdots w_n$  and  $x$  occur at the basepoint  $p$ . The representative  $\omega$  of  $w_1w_2\cdots w_n$  intersects  $x$  in a point which is a multiple self-intersection point of  $\omega$ . This does not present any difficulty in computing the bracket, because the intersection points of both curves are still transversal double points. □

## 4 HNN extensions

This section is the HNN counterpart of Section 2 and the statements, arguments and posterior use of the statements are similar. The main goal consists in proving that the products of certain cyclically reduced sequences cannot be conjugate (Theorems 4.16 and 4.19.) This result will be used to show that the pairs of terms of the bracket of certain conjugacy classes with opposite sign do not cancel.

Let  $G$  be a group, let  $A$  and  $B$  be two subgroups of  $G$  and let  $\varphi: A \longrightarrow B$  be an isomorphism. Then the *HNN extension of  $G$  relative to  $A, B$  and  $\varphi$  with stable letter  $t$*  (or, more briefly, the *HNN extension of  $G$  relative to  $\varphi$* ) will be denoted by  $G^{*\varphi}$  and is the group obtained by taking the quotient of the free product of  $G$  and the free group generated by  $t$  by the normal subgroup generated by  $t^{-1}at\varphi(a)^{-1}$  for all  $a \in A$ . (see [25] for detailed definitions.)

**Definition 4.1.** Consider an HNN extension  $G^{*\varphi}$ . Let  $n$  be a non-negative integer and for each  $i \in \{1, 2, \dots, n\}$ , let  $\varepsilon_i \in \{1, -1\}$  and  $g_i$  be an element of  $G$ . A finite sequence  $(g_0, t^{\varepsilon_1}, g_1, t^{\varepsilon_2}, \dots, g_{n-1}, t^{\varepsilon_n}, g_n)$  is said to be *reduced* if there is no consecutive subsequence of the form  $(t^{-1}, g_i, t)$  with  $g_i \in A$  or  $(t, g_j, t^{-1})$  with  $g_j \in B$ . □

The following result is the analogue of Theorem 2.2 for HNN extensions (see [25] or [8].)

**Theorem 4.2.** (1) (*Britton's lemma*) If the sequence  $(g_0, t^{\varepsilon_1}, g_1, t^{\varepsilon_2}, \dots, g_{n-1}, t^{\varepsilon_n}, g_n)$  is reduced and  $n \geq 1$  then the product  $g_0t^{\varepsilon_1}g_1t^{\varepsilon_2}\cdots g_{n-1}t^{\varepsilon_n}g_n$  is not the identity in the HNN extension  $G^{*\varphi}$ .

(2) Every element  $g$  of  $G^{*\varphi}$  can be written as a product  $g_0t^{\varepsilon_1}g_1t^{\varepsilon_2}\cdots g_{n-1}t^{\varepsilon_n}g_n$  where the sequence  $(g_0, t^{\varepsilon_1}, g_1, t^{\varepsilon_2}, \dots, g_{n-1}, t^{\varepsilon_n}, g_n)$  is reduced.

As in the case of Theorem 2.3, we include the proof of the next known result because we were unable to find it in the literature.

**Theorem 4.3.** *Suppose that the equality*

$$g_0 t^{\varepsilon_1} g_1 t^{\varepsilon_2} \cdots g_{n-1} t^{\varepsilon_n} g_n = h_0 t^{\eta_1} h_1 t^{\eta_2} \cdots h_{n-1} t^{\eta_n} h_n$$

*holds where  $(g_0, t^{\varepsilon_1}, g_1, t^{\varepsilon_2}, \dots, g_{n-1}, t^{\varepsilon_n}, g_n)$  and  $(h_0, t^{\eta_1}, h_1, t^{\eta_2}, \dots, h_{n-1}, t^{\eta_n}, h_n)$  are reduced sequences.*

*Then for each  $i \in \{1, 2, \dots, n\}$  we have that  $\varepsilon_i = \eta_i$ . Moreover, there exists a sequence of elements  $(c_1, c_2, \dots, c_n)$  in  $A \cup B$  such that*

- (1)  $g_0 = h_0 c_1$
- (2)  $g_n = \varphi^{\varepsilon_n}(c_n^{-1}) h_n$
- (3) For each  $i \in \{1, 2, \dots, n-1\}$ ,  $g_i = \varphi^{\varepsilon_i}(c_i^{-1}) h_i c_{i+1}$
- (4) For each  $i \in \{1, 2, \dots, n\}$ ,  $c_i \in A$  if  $\varepsilon_i = 1$  and  $c_i \in B$  if  $\varepsilon_i = -1$ .

*Proof.* If the two products are equal then

$$h_n^{-1} t^{-\eta_n} \cdots t^{-\eta_2} h_1^{-1} t^{-\eta_1} h_0^{-1} g_0 t^{\varepsilon_1} g_1 t^{\varepsilon_2} \cdots g_{n-1} t^{\varepsilon_n} g_n = 1 \quad (4.1)$$

By Theorem 4.2(1) the sequence that yields the product on the left hand side of Equation (4.1) is not reduced. This implies that  $\varepsilon_1 = \eta_1$ . Moreover, if  $\varepsilon_1 = 1$  then  $h_0^{-1} g_0 \in A$  and if  $\varepsilon_1 = -1$  then  $h_0^{-1} g_0 \in B$ .

Denote the product  $h_0^{-1} g_0$  by  $c_1$ . Thus,  $g_0 = h_0 c_1$ . By definition of HNN extension, we can replace  $t^{-\varepsilon_1} c_1 t^{\varepsilon_1}$  by  $\varphi^{\varepsilon_1}(c_1)$  in Equation (4.1) to obtain,

$$h_n^{-1} t^{-\eta_n} \cdots t^{-\eta_2} h_1^{-1} \varphi^{\varepsilon_1}(c_1) g_1 t^{\varepsilon_2} \cdots g_{n-1} t^{\varepsilon_n} g_n = 1 \quad (4.2)$$

By Theorem 4.2, the sequence yielding the product of the left hand side of Equation (4.2) is not reduced. Hence,  $\varepsilon_2 = \eta_2$  and if  $\varepsilon_2 = 1$  then  $h_1^{-1} \varphi^{\varepsilon_1}(c_1) g_1 \in A$  and if  $\varepsilon_2 = -1$  then  $h_1^{-1} \varphi^{\varepsilon_1}(c_1) g_1 \in B$ .

Denote by  $c_2$  the product  $h_1^{-1} \varphi^{\varepsilon_1}(c_1) g_1$ . Thus,  $g_1 = \varphi^{\varepsilon_1}(c_1^{-1}) h_1 c_2$ .

By applying these arguments, we can complete the proof by induction. ■

**Definition 4.4.** Let  $n$  be a non-negative integer. A sequence of elements of  $G^{*\varphi}$ ,  $(g_0, t^{\varepsilon_1}, g_1, t^{\varepsilon_2}, \dots, g_{n-1}, t^{\varepsilon_n})$  is said to be *cyclically reduced* if all its cyclic permutations of are reduced. □

We could not find a direct proof in the literature of the first statement of our next so we include it here. (The second statement also follows from Theorem 4.7 but it is a direct consequence of our proof.)



**Theorem 4.5.** *Let  $s$  be a conjugacy class of  $G^{*\varphi}$ . Then there exists a cyclically reduced sequence such that the product of its terms is a representative of  $s$ . Moreover, every cyclically reduced sequence with product in  $s$  has the same number of terms.*

*Proof.* If  $s$  has a representative in  $G$ , the result follows directly. So we can assume that  $s$  has no representatives in  $G$ .

By Theorem 4.2(2), the set of reduced sequences with product in  $s$  is not empty. Thus, it is possible to choose among all such sequences, one that makes the number of terms the smallest possible. Let  $(g_0, t^{\varepsilon_1}, g_1, t^{\varepsilon_2}, \dots, g_{m-1}, t^{\varepsilon_m})$  be such a sequence. We claim that  $(g_0, t^{\varepsilon_1}, g_1, t^{\varepsilon_2}, \dots, g_{m-1}, t^{\varepsilon_m})$  is cyclically reduced.

Indeed, if  $m \in \{0, 1\}$ , the sequence has the form  $(g_0)$  or  $(g_0, t^{\varepsilon_1})$  and so it is cyclically reduced. Assume now that  $m > 1$ .

If  $(g_0, t^{\varepsilon_1}, g_1, t^{\varepsilon_2}, \dots, g_{m-1}, t^{\varepsilon_m})$  is not cyclically reduced then one of the following statements holds:

- (1)  $\varepsilon_1 = 1, \varepsilon_m = -1$  and  $g_0 \in A$ .
- (2)  $\varepsilon_1 = -1, \varepsilon_m = 1$  and  $g_0 \in B$ .

We prove the result in case (1). (Case (2) can be treated with similar ideas.) In this case,  $t^{-\varepsilon_m} g_0 t^{\varepsilon_1} = \varphi(g_0) \in B$

The sequence  $(g_{m-1} \varphi(g_0) g_1, t^{\varepsilon_2}, \dots, g_{m-2}, t^{\varepsilon_{m-1}})$  is reduced, has product in  $s$  and has strictly fewer terms than the sequence  $(g_0, t^{\varepsilon_1}, g_1, t^{\varepsilon_2}, \dots, g_{m-1}, t^{\varepsilon_m})$ . This contradicts our assumption that our original sequence has the smallest number of terms. Thus, our proof is complete. ■

**Notation 4.6.** By definition, the cyclically reduced sequences of given HNN extension have the form  $(g_0, t^{\varepsilon_1}, g_1, t^{\varepsilon_2}, \dots, g_{m-1}, t^{\varepsilon_m})$ . From now on, we will make use of the following convention: For every integer  $h$ ,  $g_h$  will denote  $g_i$  where  $i$  is the unique integer in  $\{0, 1, 2, \dots, n-1\}$  such that  $n$  divides  $i - h$ . Analogously,  $\varepsilon_h$  will denote  $\varepsilon_i$  where  $i$  is the unique integer in  $\{1, 2, \dots, n\}$  such that  $n$  divides  $i - h$ . □

The next result is due to Collins and gives necessary conditions for two cyclically reduced sequence have conjugate product(see [25].)

**Theorem 4.7.** (*Collins' Lemma*) *Let  $n \geq 1$  and let  $(g_0, t^{\varepsilon_1}, g_1, t^{\varepsilon_2}, \dots, g_{n-1}, t^{\varepsilon_n})$  and  $(h_0, t^{\eta_1}, h_1, t^{\eta_2}, \dots, h_{m-1}, t^{\eta_m})$  be two cyclically reduced sequences such that their products are conjugate. Then  $n = m$  and there exist  $c \in A \cup B$  and  $k \in \{0, 1, 2, \dots, n-1\}$  such that the following holds:*

- (1)  $\eta_k = \varepsilon_n$ ,

- (2)  $c \in A$  if  $\varepsilon_n = -1$  and  $c \in B$  if  $\varepsilon_n = 1$ ,  
(3)  $g_0 t^{\varepsilon_1} g_1 t^{\varepsilon_2} \dots g_{n-1} t^{\varepsilon_n} = c^{-1} h_k t^{\eta_{k+1}} h_{k+1} t^{\eta_{k+2}} \dots h_{k+n-1} t^{\eta_k} c$ .

By Theorems 4.3 and 4.7 and arguments exactly like those of Theorem 2.8, we obtain the following result.

**Theorem 4.8.** *Let  $(g_0, t^{\varepsilon_1}, g_1, t^{\varepsilon_2}, \dots, g_{n-1}, t^{\varepsilon_n})$  and  $(h_0, t^{\eta_1}, h_1, t^{\eta_2}, \dots, h_{n-1}, t^{\eta_n})$  be cyclically reduced sequences such that the products*

$$g_0 t^{\varepsilon_1} g_1 t^{\varepsilon_2} \dots g_{n-1} t^{\varepsilon_n} \text{ and } h_0 t^{\eta_1} h_1 t^{\eta_2} \dots h_{n-1} t^{\eta_n}$$

*are conjugate. Moreover, assume that  $n \geq 1$ . Then there exists an integer  $k$  such that for each  $i \in \{1, 2, \dots, n\}$ ,  $\varepsilon_i = \eta_{i+k}$ . Moreover, there exists a sequence of elements  $(c_1, c_2, \dots, c_n)$  in  $A \cup B$  such that for each  $i \in \{1, 2, \dots, n\}$ ,  $c_i \in A$  if  $\varepsilon_i = 1$  and  $c_i \in B$  if  $\varepsilon_i = -1$  and*

$$g_i = \varphi^{\eta_{i+k}}(c_{i+k}^{-1}) h_{i+k} c_{i+k+1}$$

**Notation 4.9.** Let  $G^{*\varphi}$  be an HNN extension, where  $\varphi: A \rightarrow B$ . We denote the subgroup  $A$  by  $C_1$  and the subgroup  $B$  by  $C_{-1}$ .  $\square$

**Corollary 4.10.** *Let  $(g_0, t^{\varepsilon_1}, g_1, t^{\varepsilon_2}, \dots, g_{n-1}, t^{\varepsilon_n})$  and  $(h_0, t^{\eta_1}, h_1, t^{\eta_2}, \dots, h_{n-1}, t^{\eta_n})$  be cyclically reduced sequences such that the products*

$$g_0 t^{\varepsilon_1} g_1 t^{\varepsilon_2} \dots g_{n-1} t^{\varepsilon_n} \text{ and } h_0 t^{\eta_1} h_1 t^{\eta_2} \dots h_{n-1} t^{\eta_n}$$

*are conjugate. If  $n \geq 1$  then there exists an integer  $k$  such that for each  $i \in \{1, 2, \dots, n\}$ ,  $\varepsilon_i = \eta_{i+k}$  and  $g_i$  belongs to the double coset  $C_{-\varepsilon_i} h_{i+k} C_{\varepsilon_{i+1}}$ .*

**Remark 4.11.** If  $\varepsilon \in \{1, -1\}$  and  $a \in C_\varepsilon$  then  $at^\varepsilon = t^\varepsilon \varphi^\varepsilon(a)$  in  $G^{*\varphi}$ .  $\square$

**Definition 4.12.** Let  $n \geq 2$  and let  $(g_0, t^{\varepsilon_1}, g_1, t^{\varepsilon_2}, \dots, g_{n-1}, t^{\varepsilon_n})$  be a cyclically reduced sequence. The *sequence of double cosets associated with  $(g_0, t^{\varepsilon_1}, g_1, t^{\varepsilon_2}, \dots, g_{n-1}, t^{\varepsilon_n})$*  is the sequence of double cosets

$$(C_{\varepsilon_i} t^{\varepsilon_i} g_i t^{\varepsilon_{i+1}} g_{i+1} t^{\varepsilon_{i+2}} C_{-\varepsilon_{i+2}})_{0 \leq i \leq n-1}.$$

$\square$

**Lemma 4.13.** *Let  $(g_0, t^{\varepsilon_1}, g_1, t^{\varepsilon_2}, \dots, g_{n-1}, t^{\varepsilon_n})$  and  $(h_0, t^{\eta_1}, h_1, t^{\eta_2}, \dots, h_{n-1}, t^{\eta_n})$  be two cyclically reduced sequences whose products are conjugate and such that  $n \geq 2$ . Then the sequence of double cosets associated with  $(g_0, t^{\varepsilon_1}, g_1, t^{\varepsilon_2}, \dots, g_{n-1}, t^{\varepsilon_n})$  is a cyclic permutation of the cyclic of double cosets associated with  $(h_0, t^{\eta_1}, h_1, t^{\eta_2}, \dots, h_{n-1}, t^{\eta_n})$ .*

*Proof.* Let  $k \in \{1, 2, \dots, n\}$  and  $(c_1, c_2, \dots, c_n)$  be a finite sequence of elements in  $A \cup B$  given by Theorem 4.8 for the sequences  $(g_0, t^{\varepsilon_1}, g_1, t^{\varepsilon_2}, \dots, g_{n-1}, t^{\varepsilon_n})$  and  $(h_0, t^{\eta_1}, h_1, t^{\eta_2}, \dots, h_{n-1}, t^{\eta_n})$ . By cyclically rotating  $(h_0, t^{\eta_1}, h_1, t^{\eta_2}, \dots, h_{n-1}, t^{\eta_n})$  if necessary, we can assume that  $k = 0$ . (We could carry out the proof with  $k > 0$ , but the assumption  $k = 0$  makes the equations more neat).

Let  $i \in \{1, 2, \dots, n\}$ . We will complete this proof by showing that the  $i$ -th double coset of the sequence  $(g_0, t^{\varepsilon_1}, g_1, t^{\varepsilon_2}, \dots, g_{n-1}, t^{\varepsilon_n})$  equals the  $i$ -th double coset of the sequence of  $(h_0, t^{\eta_1}, h_1, t^{\eta_2}, \dots, h_{n-1}, t^{\eta_n})$ .

By Theorem 4.8,  $\varepsilon_j = \eta_j$  for each  $j \in \{1, 2, \dots, n\}$  and

$$g_i t^{\varepsilon_{i+1}} g_{i+1} = \varphi^{\eta_i}(c_i^{-1}) h_i c_{i+1} t^{\eta_{i+1}} \varphi^{\eta_{i+1}}(c_{i+1}^{-1}) h_{i+1} c_{i+2}$$

By Remark 4.11 we have that  $c_{i+1} t^{\eta_{i+1}} \varphi^{\eta_{i+1}}(c_{i+1}^{-1}) = t^{\eta_{i+1}}$ . Thus,

$$g_i t^{\varepsilon_{i+1}} g_{i+1} = \varphi^{\eta_i}(c_i^{-1}) h_i t^{\eta_{i+1}} h_{i+1} c_{i+2}$$

By Theorem 4.8,  $c_i \in C_{\varepsilon_i}$  and  $c_{i+2} \in C_{\varepsilon_{i+2}}$ . Therefore,  $\varphi^{\eta_i}(c_i^{-1}) \in C_{-\varepsilon_i}$ . Thus,

$$t^{\varepsilon_i} g_i t^{\varepsilon_{i+1}} g_{i+1} t^{\varepsilon_{i+2}} \in t^{\varepsilon_i} C_{-\varepsilon_i} h_i t^{\eta_{i+1}} h_{i+1} C_{\varepsilon_{i+2}} t^{\varepsilon_i}$$

Consequently, by Remark 4.11,  $t^{\varepsilon_i} g_i t^{\varepsilon_{i+1}} g_{i+1} t^{\varepsilon_{i+2}}$  is in the  $i$ -th double coset associated with  $(h_0, t^{\eta_1}, h_1, t^{\eta_2}, \dots, h_{n-1}, t^{\eta_n})$  and our proof is complete.  $\blacksquare$

**Definition 4.14.** Let  $G^{*\varphi}$  be an HNN extension, where  $\varphi: A \longrightarrow B$ . We say that  $G^{*\varphi}$  is *separated* if  $A \cap B^g = \{1\}$  for all  $g \in G$ .  $\square$

**Lemma 4.15.** Let  $(t^{\varepsilon_1}, g_1, t^{\varepsilon_2}, g_2, t^{\varepsilon_3})$  and  $(t^{\eta_1}, h_1, t^{\eta_2}, h_2, t^{\eta_3})$  be two reduced sequences. Let  $a \in C_{\varepsilon_2}$  and let  $b \in C_{\eta_2}$ . Suppose that  $G^{*\varphi}$  is separated and that  $A$  and  $B$  are malnormal in  $G^{*\varphi}$ . Then the following statements hold.

- (1) If the double cosets  $C_{\varepsilon_1} t^{\varepsilon_1} g_1 a t^{\varepsilon_2} g_2 t^{\varepsilon_3} C_{-\varepsilon_3}$  and  $C_{\varepsilon_1} t^{\varepsilon_1} g_1 t^{\varepsilon_2} g_2 t^{\varepsilon_3} C_{-\varepsilon_3}$  are equal then  $a = 1$ .
- (2) If the subsets of double cosets  $\{C_{\varepsilon_1} t^{\varepsilon_1} g_1 a t^{\varepsilon_2} g_2 t^{\varepsilon_3} C_{-\varepsilon_3}, C_{\eta_1} t^{\eta_1} h_1 t^{\eta_2} h_2 t^{\eta_3} C_{\varepsilon_3}\}$  and  $\{C_{\varepsilon_1} t^{\varepsilon_1} g_1 t^{\varepsilon_2} g_2 t^{\varepsilon_3} C_{-\varepsilon_3}, C_{\eta_1} t^{\eta_1} h_1 b t^{\eta_2} h_2 t^{\eta_3} C_{\varepsilon_3}\}$  are equal then  $a$  and  $b$  are conjugate by an element of  $A \cup B$ . Moreover, if  $a \neq 1$  or  $b \neq 1$ , then  $\varepsilon_2 = \eta_2$ .

*Proof.* We first prove (1). By Remark 4.11,

$$t^{\varepsilon_1} C_{-\varepsilon_1} g_1 t^{\varepsilon_2} g_2 C_{\varepsilon_3} t^{\varepsilon_3} = t^{\varepsilon_1} C_{-\varepsilon_1} g_1 t^{\varepsilon_2} g_2 C_{\varepsilon_3} t^{\varepsilon_3}.$$

Now we cross out  $t^{\varepsilon_1}$  and  $t^{\varepsilon_3}$  at both sides of the above equation. From the equality we obtain we deduce that there exist  $d_1 \in C_{\varepsilon_1}$  and  $d_2 \in C_{\varepsilon_3}$  such that  $g_0 t^{\varepsilon_2} g_1 =$

$d_1 g_0 a t^{\varepsilon_2} g_1 d_2$ . By Theorem 4.3, there exists  $c \in C_{\varepsilon_2}$  such that

$$g_0 = d_1 g_0 a c \quad (4.3)$$

$$g_1 = \varphi^{\varepsilon_2}(c^{-1}) g_1 d_2. \quad (4.4)$$

By malnormality, separability, and Equation (4.4),  $\varphi^{\varepsilon_2}(c^{-1}) = 1$  and so  $c = 1$ . By malnormality, separability and Equation (4.3),  $a c = 1$ . Hence,  $a = 1$ .

Now we prove (2). Observe that if the two sets of the statement of the lemma are equal, then there are two possibilities:

- (i) The first (resp. second) element listed in the right set is equal to the first (resp. second) element listed on the second set.
- (ii) The first (resp. second) element listed in the right set is equal to the second (resp. first) element listed on the second set.

The case (i) is ruled out by statement (1). Hence, we can assume that (ii) holds. By Theorem 4.3, for each  $i \in \{1, 2, 3\}$ ,  $\varepsilon_i = \eta_i$ . By arguing as in the proof of statement (1), we can deduce that there exist  $c_1$  and  $d_1$  in  $C_{-\varepsilon_1}$  and  $c_3$  and  $d_3$  in  $C_{\varepsilon_3}$ , such that

$$g_1 a t^{\varepsilon_2} g_2 = c_1 h_1 b t^{\varepsilon_2} h_2 c_3 \text{ and } g_1 t^{\varepsilon_2} g_2 = d_1 h_1 t^{\varepsilon_2} h_2 d_3$$

By Theorem 4.3, there exist a pair of elements  $x$  and  $y$  in  $C_{\varepsilon_2}$  such that

$$g_0 a = c_1 h_0 b x \quad (4.5)$$

$$g_1 = \varphi^{\varepsilon_2}(x^{-1}) h_1 c_3. \quad (4.6)$$

$$g_0 = d_1 h_0 y \quad (4.7)$$

$$g_1 = \varphi^{\varepsilon_2}(y^{-1}) h_1 d_3. \quad (4.8)$$

Since  $\varphi$  is an isomorphism, by malnormality and separability and Equations (4.6) and (4.8),  $x = y$ . Analogously, by malnormality and separability and Equations (4.5) and (4.7),  $b x a^{-1} = y$ . Therefore,  $a$  and  $b$  are conjugate by  $x$ . Since  $x \in A \cup B$ , the proof is complete. ■

The next theorem gives necessary conditions for certain products of cyclically reduced sequences to not be conjugate.

**Theorem 4.16.** *Let  $G^{*\varphi}$  be an HNN extension. Let  $(g_0, t^{\varepsilon_1}, g_1, t^{\varepsilon_2}, \dots, g_{n-1}, t^{\varepsilon_n})$  be a cyclically reduced sequence. and let  $i, j \in \{1, 2, \dots, n\}$ . Let  $a$  be an element of  $C_{\varepsilon_i}$  and let  $b$  be an element of  $C_{\varepsilon_j}$ . Moreover, assume that the following conditions hold:*

- (1) *The subgroups  $A$  and  $B$  are malnormal in  $G$ .*

- (2) The HNN extension  $G^{*\varphi}$  is separated.
- (3) If  $\varepsilon_i = \varepsilon_j$  then  $a$  and  $b$  are not conjugate by an element of  $A \cup B$ .
- (4) Either  $a \neq 1$  or  $b \neq 1$ .

Then the products

$$g_0 t^{\varepsilon_1} g_1 t^{\varepsilon_2} \cdots g_{i-1} a t^{\varepsilon_i} g_i \cdots g_{n-1} t^{\varepsilon_n} \text{ and } g_0 t^{\varepsilon_1} g_1 t^{\varepsilon_2} \cdots g_{j-1} b t^{\varepsilon_j} g_j \cdots g_{n-1} t^{\varepsilon_n}$$

are not conjugate.

*Proof.* Assume that the products of the hypothesis of the theorem are conjugate. Furthermore, we assume that  $a$  is not the identity. The case of  $b$  not the identity can be treated similarly.

If  $n = 1$  the proof of the result is direct. Hence, we can assume  $n \geq 2$ .

The sequences

$$(g_0, t^{\varepsilon_1}, g_1, t^{\varepsilon_2}, \dots, g_i a, t^{\varepsilon_{i+1}}, \dots, g_{n-1}, t^{\varepsilon_n}) \text{ and } (g_0, t^{\varepsilon_1}, g_1, t^{\varepsilon_2}, \dots, g_j b, t^{\varepsilon_{j+1}}, \dots, g_{n-1}, t^{\varepsilon_n})$$

are cyclically reduced. By Lemma 4.13 both sequences are associated with the same cyclic sequence of double cosets. If  $i = j$  the result is a consequence of Lemma 4.15(1). Hence, we can assume that  $i \neq j$ . As in the proof of Theorem 2.13 by crossing out the elements in the sequences of double cosets with equal expressions we obtain the two sets below are equal,

$$\begin{aligned} &\{C_{\varepsilon_{i-1}} t^{\varepsilon_{i-1}} g_{i-1} a t^{\varepsilon_i} g_i t^{\varepsilon_{i+1}} C_{-\varepsilon_{i+1}}, C_{\varepsilon_{j-1}} t^{\varepsilon_{j-1}} g_{j-1} t^{\varepsilon_j} g_j t^{\varepsilon_{j+1}} C_{-\varepsilon_{j+1}}\} \\ &\{C_{\varepsilon_{i-1}} t^{\varepsilon_{i-1}} g_{i-1} t^{\varepsilon_i} g_i t^{\varepsilon_{i+1}} C_{-\varepsilon_{i+1}}, C_{\varepsilon_{j-1}} t^{\varepsilon_{j-1}} g_{j-1} b t^{\varepsilon_j} g_j t^{\varepsilon_{j+1}} C_{-\varepsilon_{j+1}}\} \end{aligned}$$

(By Remark 4.11,

$$C_{\varepsilon_{i-2}} t^{\varepsilon_{i-2}} g_{i-2} t^{\varepsilon_{i-1}} g_{i-1} a t^{\varepsilon_i} C_{-\varepsilon_i} = C_{\varepsilon_{i-2}} t^{\varepsilon_{i-2}} g_{i-2} t^{\varepsilon_{i-1}} g_{i-1} t^{\varepsilon_i} C_{-\varepsilon_i}.$$

Hence, double cosets like those on the left hand side of the above equation do not appear in the list of possibly distinct double cosets). By Lemma 4.15,  $\varepsilon_i = \varepsilon_j$  and  $a$  and  $b$  are conjugate by an element of  $A \cup B$ . This contradicts our hypothesis and so the proof is complete. ■

**Remark 4.17.** The following example shows that hypotheses (1) or (2) of Theorem 4.16 are necessary. Let  $G$  be the direct sum of  $\mathbb{Z}$  and  $\mathbb{Z}$ ,  $\mathbb{Z} \oplus \mathbb{Z}$ . Let  $A = \mathbb{Z} \oplus \{0\}$  and let  $B = \{0\} \oplus \mathbb{Z}$  considered as subgroups of  $\mathbb{Z}$ . Define  $\varphi: A \rightarrow B$  by  $\varphi(x, 0) = (0, x)$ . Let  $a$  be an element of  $A$ . Since  $\mathbb{Z} \oplus \mathbb{Z}$  is commutative,

$$(0, 1)t^{-1}(1, 1)t^{-1}(1, 1)t(0, -1) = t^{-1}(2, 1)t^{-1}(0, 1)t$$

Thus the sequences  $t^{-1}, (1, 1), t^{-1}, (0, 1)(1, 0), t$  and  $t^{-1}, (1, 1)(1, 0), t^{-1}, (0, 1), t$  have conjugate products. On the other hand, hypotheses (3) and (4) of Theorem 4.16 hold for these sequences.

The following example shows that hypothesis (3) of Theorem 4.16 is necessary. Let  $G^*$  be an HNN extension relative to an isomorphism  $\varphi: A \longrightarrow B$ . Let  $g \in G \setminus A$  and let  $a \in A$ . The sequence  $(g, t, g, t)$  is cyclically reduced but the product of the sequences  $(ga, t, g, t)$  and  $(g, t, ga, t)$  are conjugate.

□

The next auxiliary lemma will be used in the proof of Theorem 4.19. The set of congruence classes modulo  $n$  is denoted by  $\mathbb{Z}/n\mathbb{Z}$ .

**Lemma 4.18.** *Let  $n$  and  $k$  be integers. Let  $\widehat{F}: \mathbb{Z}/n\mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z}$  be induced by the map on the integers  $F(x) = -x + k$ . If  $n$  is odd or  $k$  is even then  $\widehat{F}$  has a fixed point.*

*Proof.* If  $k$  is even then  $\frac{k}{2}$  is integer and a fixed point of  $F$ . Thus  $\widehat{F}$  has a fixed point.

On the other hand, the map  $\widehat{F}$  has a fixed point whenever the equation  $2x \equiv -k \pmod{n}$  has a solution. If  $n$  is odd this equation has a solution because 2 has an inverse in  $\mathbb{Z}/n\mathbb{Z}$ . This completes the proof. ■

We will use Notation 4.9 for the statement and proof of the next result.

**Theorem 4.19.** *Let  $G^{*\varphi}$  be an HNN extension. Let  $(g_0 t^{\varepsilon_1} g_1 t^{\varepsilon_2} \dots g_{n-1} t^{\varepsilon_n})$  be a cyclically reduced sequence. Let  $i$  and  $j$  be elements of  $\{1, 2, \dots, n\}$ . Let  $a$  be an element of  $C_{\varepsilon_i}$  and let  $b$  an element of  $C_{-\varepsilon_j}$ . Assume that for each  $g \in G$ ,  $g^{-1}$  does not belong to the set  $(A \cup B)g(A \cup B)$ . Then the products*

$$g_0 t^{\varepsilon_1} g_1 t^{\varepsilon_2} \dots g_{i-1} a t^{\varepsilon_i} g_i \dots g_{n-1} t^{\varepsilon_n} \text{ and } g_{n-1}^{-1} t^{-\varepsilon_{n-1}} g_{n-2}^{-1} t^{-\varepsilon_{n-2}} \dots g_j^{-1} b t^{-\varepsilon_j} \dots t^{-\varepsilon_1} g_0^{-1} t^{-\varepsilon_n}$$

*are not conjugate.*

*Proof.* We start by giving a sketch of the proof: If the above products are conjugate then the sequence of  $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$  is a rotation of the sequence  $(-\varepsilon_n, -\varepsilon_{n-1}, \dots, -\varepsilon_1)$ . Since the terms of those sequences are not zero, a term of the first of the form  $\varepsilon_h$  cannot correspond to a term of the form  $-\varepsilon_h$ . This gives conditions of the rotation and the number of terms. The same rotation also establishes a correspondence between double cosets of  $g_h$ 's and double cosets of rotated  $g_h^{-1}$ 's. Using the fact that the sequences  $\varepsilon_h$ 's and  $g_h$ 's are “off” by one, we will show that there exists  $u$  such that  $g_u^{-1} \in$

$(A \cup B)g_u(A \cup B)$ . (in the detailed proof, we need to study separately the cases where the elements  $a$  and  $b$  appear in the equations.)

Here is the detailed proof. Consider the sequence  $(s_0, t^{\eta_1}, s_1, t^{\eta_2}, \dots, s_{n-1}, t^{\eta_n})$  defined by  $\eta_h = -\varepsilon_{n-h}$  and

$$s_h = \begin{cases} g_{-h-1}^{-1}b & \text{if } -h-1 \equiv j \pmod{n}, \\ g_{-h-1}^{-1} & \text{otherwise.} \end{cases} \quad (4.9)$$

for each  $h$ . (Recall Notation 4.6.)

Assume that the products of the hypothesis of the theorem are conjugate. Thus the sequences

$$(g_0, t^{\varepsilon_1}, g_1, t^{\varepsilon_2}, \dots, g_{i-1}a, t^{\varepsilon_i}, g_i, \dots, g_{n-1}, t^{\varepsilon_n}) \text{ and } (s_0, t^{\eta_1}, s_1, t^{\eta_2}, \dots, s_{n-1}, t^{\eta_n})$$

are cyclically reduced and have conjugate products. Let  $k$  be as in Corollary 4.10 for these two products. Hence,

$$\varepsilon_h = \eta_{h+k} = -\varepsilon_{n-h-k}, \quad (4.10)$$

and the following holds:

- (1) if  $h \not\equiv i-1 \pmod{n}$  then  $g_h$  belongs to the double coset  $C_{-\varepsilon_h}s_{h+k}C_{\varepsilon_{h+1}}$
- (2)  $g_{i-1}a$  belongs to the double coset  $C_{-\varepsilon_{i-1}}s_{i+k}C_{\varepsilon_i}$ .

By hypothesis,  $a \in C_{\varepsilon_i}$ . Thus, for all  $h$  we have

$$g_h \in C_{-\varepsilon_h}s_{h+k}C_{\varepsilon_{h+1}} \quad (4.11)$$

Since for every integer  $h$ ,  $\varepsilon_h \neq 0$  we have that  $\varepsilon_h \neq -\varepsilon_h$ . Therefore, by Equation (4.10), the map  $\hat{F}$  defined on the integers mod  $n$  by the formula  $F(h) = -h-k$  cannot have fixed points. By Lemma 4.18,  $n$  is even and  $k$  is odd.

Then  $(-k-1)$  is even. By Lemma 4.18, the map  $G(h) = -h + (-k-1)$  has a fixed point. Denote this fixed point by  $u$ . Thus

$$u+k \equiv -u-1 \pmod{n} \quad (4.12)$$

Assume that  $u \not\equiv j \pmod{n}$ . By Equations (4.12) and (4.9),  $s_{u+k} = s_{-u-1} = g_u^{-1}$ . By Equation 4.11,  $g_u \in C_{-\varepsilon_u}g_u^{-1}C_{\varepsilon_{u+1}} \subset (A \cup B)g_j(A \cup B)$ , contradicting our hypothesis. Therefore,  $u \equiv j \pmod{n}$ . In this case, by Equation 4.11,  $g_j \in C_{-\varepsilon_j}g_j^{-1}bC_{\varepsilon_{j+1}}$ .

By Equations (4.12) and (4.10),  $\varepsilon_{j+1} = -\varepsilon_{-j-1-k} = -\varepsilon_j$ . Since  $b \in C_{-\varepsilon_j}$  then  $b \in C_{\varepsilon_{j+1}}$ . Consequently,  $g_j \in C_{-\varepsilon_j}g_j^{-1}C_{\varepsilon_{j+1}}$ . Since  $C_{-\varepsilon_j}g_j^{-1}C_{\varepsilon_{j+1}} \subset (A \cup B)g_j(A \cup B)$ , this is a contradiction. ■

## 5 Oriented non-separating simple loops

This section is the "separating version" of Section 3. The main purpose here consists in proving Theorem 5.3, which describes the terms of the bracket of a simple non-separating conjugacy class and an arbitrary conjugacy class.

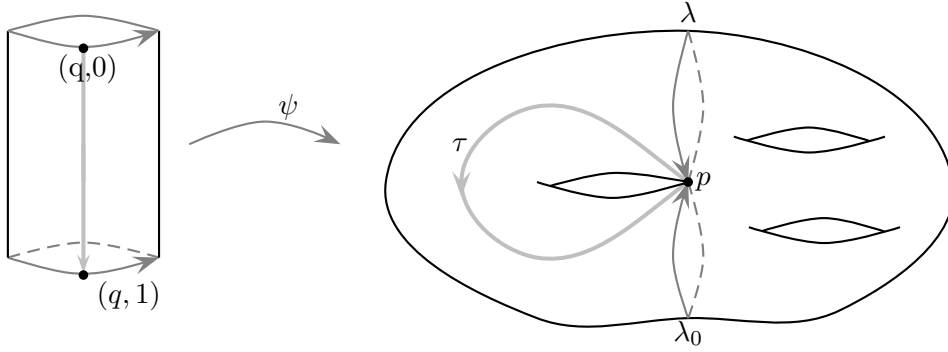


Figure 4: The map  $\psi$  of Lemma 5.1

We will start by proving some elementary auxiliary results.

**Lemma 5.1.** *Let  $\lambda$  be a non-separating simple curve on  $\Sigma$  and  $p$  a point in  $\lambda$ . There exists a map  $\psi: \mathbb{S}^1 \times [0, 1] \longrightarrow \Sigma$ , such that:*

- (1) *There exists a point  $q \in \mathbb{S}^1$ ,  $\psi(q, 0) = \psi(q, 1) = p$ .*
- (2)  *$\psi$  is injective on  $(\mathbb{S}^1 \times [0, 1]) \setminus \{(q, 0), (q, 1)\}$*
- (3)  *$\psi|_{\mathbb{S}^1 \times \{0\}} = \lambda$ .*

*Proof.* Choose a simple trivial curve  $\tau$  such that  $\tau \cap \lambda = \{p\}$  and the intersection of  $\tau$  and  $\lambda$  is transversal. (The existence of such a curve is guaranteed by the following argument of Poincaré: take a small arc  $\beta$  crossing  $\lambda$  transversally. Since  $\Sigma \setminus (\beta \cup \lambda)$  is connected there exists a non-trivial arc in  $\Sigma \setminus (\beta \cup \lambda)$ , with no self-intersections, joining the endpoints of  $\beta$ .)

Consider an injective map  $\eta: \mathbb{S}^1 \times [0, 1] \longrightarrow \Sigma$  such that  $\lambda = \eta(\mathbb{S}^1 \times \{0\})$ . Let  $q \in \mathbb{S}^1$  be such that  $\eta(q, 0) = p$ . Denote by  $C$  the image of the cylinder  $\eta(\mathbb{S}^1 \times [0, 1])$ . Denote by  $\xi$  the boundary component of  $C$  defined by  $\eta(\mathbb{S}^1 \times \{1\})$ . Modify  $\eta$  if necessary so that  $\tau$  intersects  $\xi$  in a unique double point  $s$ . (See Figure 5.)

Choose two points on  $s_1$  and  $s_2$  on  $\xi$  close to  $s$  and at both sides of  $s$ . Choose an embedded arc in the interior of  $C$ , intersecting  $\tau$  exactly once, from  $s_1$  to  $s_2$  and denote



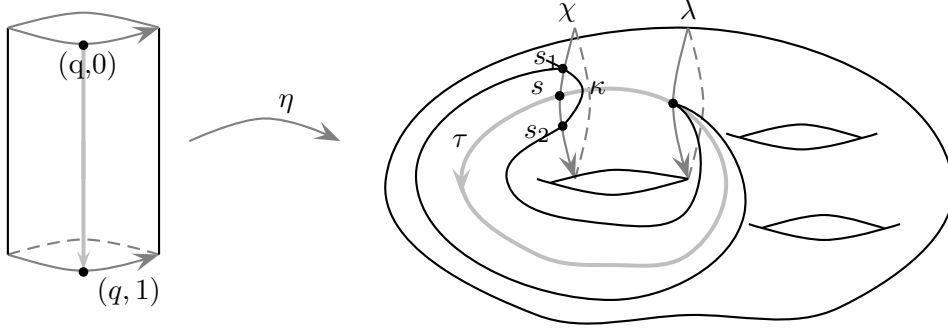


Figure 5: The proof of Lemma 5.1

it by  $\kappa$ . Denote by  $D$  the closed half disk bounded by  $\kappa$  and the subarc of  $\xi$  containing  $s$ .

Choose two disjoint embedded arcs  $\alpha_1$  and  $\alpha_2$  on  $\Sigma$  from  $p$  to  $s_1$  and  $s_2$ , respectively, and such that  $\alpha_1 \cap \tau = \alpha_2 \cap \tau = \{p\}$ ,  $\alpha_1 \cap C = \{p, s_1\}$  and  $\alpha_2 \cap C = \{p, s_2\}$ .

Consider the triangle  $T$ , with sides  $\alpha_1$ , the arc in  $\tau$  from  $s_1$  to  $s_2$  and  $\alpha_2$ .

Denote by  $\eta_1$  the restriction of  $\eta$  to  $\eta^{-1}(D)$ ,  $\eta_1: \eta^{-1}(D) \rightarrow D$ . Now, take a homeomorphism  $\eta_2: D \rightarrow D \cup T$  such that  $\eta_2|_{\kappa}$  is the identity and  $\eta_2(s) = p$ .

For each  $x \in \mathbb{S}^1 \times [0, 1]$ , define  $\psi: \mathbb{S}^1 \times [0, 1] \rightarrow \Sigma$  as  $\eta(x)$  if  $x \notin D$  and as  $\eta_2\eta(x)$  otherwise. This map satisfies the required properties. ■

Let  $\psi$  be the map of Lemma 5.1. Denote by  $\lambda_1$  the curve  $\psi(\mathbb{S}^1 \times \{0\})$ . The homeomorphism  $\vartheta: \lambda = \psi(\mathbb{S}^1 \times \{0\}) \rightarrow \lambda_1 = \psi(\mathbb{S}^1 \times \{1\})$  defined by  $\vartheta(\psi(s, 0)) = \psi(s, 1)$  induces an isomorphism  $\varphi: \pi_1(\lambda, p) \rightarrow \pi_1(\lambda_1, p)$ . Denote by  $\Sigma_1$  the subspace of  $\Sigma$  defined by  $\Sigma \setminus \psi(\mathbb{S}^1 \times (0, 1))$ , and by  $\tau$  the simple closed curve induced by the restriction of  $\psi$  to  $\{q\} \times [0, 1]$ .

**Lemma 5.2.** *With the above notation, the fundamental group of  $\Sigma$ ,  $\pi_1(\Sigma, p)$ , is isomorphic to the HNN extension of  $\pi_1(\Sigma_1, p)$  relative to  $\varphi$ . Moreover,  $\tau$  is a representative of the element denoted by the stable letter  $t$  and if  $(g_0, t^{\varepsilon_1}, g_1, t^{\varepsilon_2}, \dots, g_{n-1}, t^{\varepsilon_n})$  is a cyclically reduced sequence of the HNN extension then there exists a sequence of curves  $(\gamma_0, \gamma_1, \dots, \gamma_{n-1})$  such that for each  $i \in \{0, 1, \dots, n-1\}$ ,*

(1) *The basepoint of  $\gamma_i$  is  $p$ .*

- (2) The curve  $\gamma_i$  is a representative of  $g_i$ .
- (3) The inclusion  $\gamma_i \subset \Sigma_1$  holds.
- (4) The basepoint  $p$  is not self-intersection point of  $\gamma_i$ . In other words,  $\gamma_i$  passes through  $p$  exactly once.

*Proof.* By the van Kampen's Theorem (see, for instance [16]), since  $\Sigma_1 \cap \tau = \{p\}$  we have that  $\pi_1(\Sigma_1 \cup \tau, p)$  is the free product of  $\pi_1(\Sigma_1, p)$  and the infinite cyclic group  $\pi_1(\tau, p)$ .

Denote by  $D$  the disk  $\psi((\mathbb{S}^1 \setminus q) \times (0, 1))$ . Glue the boundary of  $D$  to the boundary of  $\Sigma_1 \cup \tau$  as follows: attach  $\lambda$ ,  $\tau$ ,  $\lambda_1$  to  $\mathbb{S}^1 \times \{0\}$ ,  $q \times [0, 1]$ , and  $\mathbb{S}^1 \times \{1\}$  respectively. (The reader can easily deduce the orientations.)

The relation added by attaching the disk  $D$  shows that the  $\pi_1(\Sigma, p)$  is isomorphic to the HNN extension of  $\pi_1(\Sigma_1, p)$  relative to  $\varphi$ . Notice also that  $\tau$  is a representative of  $t$ .

For each  $i \in \{0, 1, \dots, n-1\}$ , let  $\gamma_i$  be a loop in  $\Sigma_1$ , based at  $p$  and representing  $g_i$ . By modifying these curves by a homotopy relative to  $p$  if necessary, we can assume that each of them intersects  $p$  exactly once, as desired. Then (4) follows.  $\blacksquare$

Recall that there is a canonical isomorphism between free homotopy classes of curves on a surface  $\Sigma$  and conjugacy classes of elements of  $\pi_1(\Sigma)$ . From now on, we will identify these two sets.

The following theorem gives a combinatorial description of the bracket of two oriented curves, one of them simple and non-separating.

**Theorem 5.3.** *Let  $\lambda$  be a separating simple closed curve. Let  $(g_0, t^{\varepsilon_1}, g_1, t^{\varepsilon_2}, \dots, g_{n-1}, t^{\varepsilon_n})$  be a cyclically reduced sequence for the HNN extension of Lemma 5.2 determined by  $\lambda$ . Let  $y$  be the element of  $\pi_1(\Sigma, p)$  represented by  $\lambda$ . Then the following holds.*

- (1) *There exists a representative  $\eta$  of the conjugacy class of the product  $g_0 t^{\varepsilon_1} g_1 t^{\varepsilon_2} \cdots g_{n-1}$  such that  $\eta$  and  $\lambda$  intersect transversally at  $p$  with multiplicity  $n$ .*
- (2) *There exists  $s \in \{1, -1\}$  such the bracket is given by*

$$s [g_0 t^{\varepsilon_1} g_1 t^{\varepsilon_2} \cdots g_{n-1} t^{\varepsilon_n}, y] = \sum_{i : \varepsilon_i = 1} g_0 t^{\varepsilon_1} g_1 t^{\varepsilon_2} \cdots g_{i-1} y t^{\varepsilon_i} g_i \cdots g_{n-1} t^{\varepsilon_n} - \sum_{i : \varepsilon_i = -1} g_0 t^{\varepsilon_1} g_1 t^{\varepsilon_2} \cdots g_{i-1} \varphi(y) t^{\varepsilon_i} g_i \cdots g_{n-1} t^{\varepsilon_n}$$

*Proof.* Let  $(\gamma_0, \gamma_1, \dots, \gamma_{n-1})$  denote a sequence of curves obtained in Lemma 5.2 for the sequence  $(g_0, t^{\varepsilon_1}, g_1, t^{\varepsilon_2} \dots, g_{n-1}, t^{\varepsilon_n})$ .

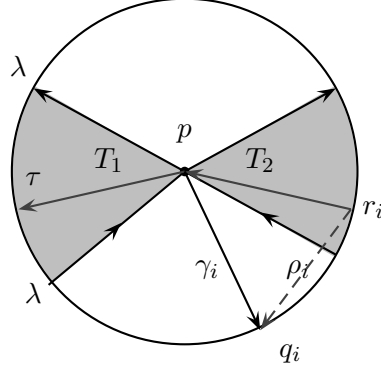


Figure 6: Proof of Theorem 5.3

Let  $D \subset \Sigma$  be a small disk around  $p$ . Observe that  $D \cap \psi(\mathbb{S}^1 \times [0, 1])$  consists in two "triangles"  $T_1$  and  $T_2$ , intersecting at  $p$  (see Figure 6.) Two of the sides of one of these triangles are subarcs of  $\lambda$ . Denote this triangle by  $T_1$ . Suppose that the beginning of  $\tau$  is inside  $T_1$  (the proof for the other possibility is analogous.)

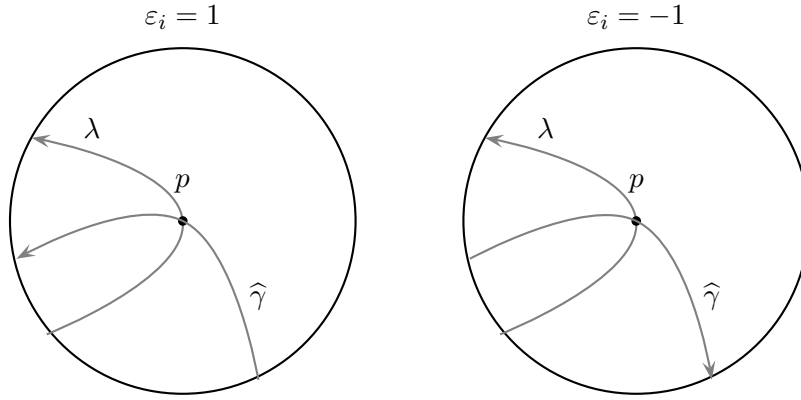


Figure 7: Proof of Theorem 5.3

For each  $i \in \{1, 2, \dots, n\}$ , if  $\varepsilon_i = 1$ ,  $\tau_i$  will denote a copy of the curve  $\tau$  and if  $\varepsilon_i = -1$ ,  $\tau_i$  will denote a copy of the curve  $\tau$  with opposite direction. Denote by  $\gamma$  the curve  $\gamma_1 \tau_1 \gamma_2 \tau_2 \dots \gamma_n \tau_n$ . Clearly,  $\gamma$  is a representative of  $g_0 t^{\varepsilon_1} g_1 t^{\varepsilon_2} \dots g_{n-1} t^{\varepsilon_n}$ .

The intersection of  $\gamma$  and  $\lambda$  consists in  $2n$  points, located at the beginning and end of  $\tau_i$  for each  $i \in \{1, 2, \dots, n\}$ .

We claim that for each  $i \in \{1, 2, \dots, n\}$  if  $\varepsilon_i = 1$  then the intersection point of  $\gamma$  and  $\lambda$  located at the end of  $\tau_i$  can be removed by a small homotopy. Similarly, if  $\varepsilon_i = -1$  then the intersection point at the beginning of  $\tau_i$  can be removed by a small homotopy.

Indeed, let  $i \in \{0, 1, \dots, n\}$  be such that  $\varepsilon_i = 1$ . The intersection  $\gamma \cap D$  contains  $2n$  subarcs of  $\gamma$ . Denote by  $\varrho$  the subarc containing the end of  $\tau_i$ . Denote by  $r_i$  the intersection of the boundary of  $D$  and the end of  $\tau_i$  (see Figure 6). Denote by  $q_i$  the intersection of the beginning of  $\gamma_i$  with the boundary of  $D$ . Choose an arc from  $r_i$  to  $q_i$  which does not intersect  $\lambda$  and denote it by  $\varrho_i$ . In  $\gamma$ , replace  $\varrho$  by  $\varrho_i$ , (see Figure 6.) This proves the claim for the case  $\varepsilon_i = 1$ . The proof of the case  $\varepsilon_i = -1$  is similar.

Denote by  $\eta$  the curve obtained after homotoping  $\gamma$  to remove the  $n$  points mentioned in the claim.

Note that  $\eta$  intersects  $\lambda$  at  $p$  with multiplicity  $n$ . More specifically, for each  $i \in \{1, 2, \dots, n\}$ , if  $\varepsilon_i = 1$  then there is an intersection at the beginning of  $\tau_i$  and if  $\varepsilon_i = -1$  there is an intersection at the end of  $\tau_i$ . Since  $\eta$  crosses  $\lambda$  these intersections are transversal. Thus, (1) is proved.

Now, we will compute the bracket  $[g_0 t^{\varepsilon_1} g_1 t^{\varepsilon_2} \cdots g_{n-1} t^{\varepsilon_n}, y]$  using  $\eta$  and  $\lambda$  as representatives. Since  $\eta$  and  $\lambda$  have  $n$  intersection points, we will find  $n$  terms.

Let  $i \in \{1, 2, \dots, n\}$ . Assume first that  $\varepsilon_i = 1$ . The term of the bracket corresponding to this intersection point is obtained by inserting  $\lambda$  between  $\gamma_{i-1}$  and  $\tau_i$ . Since the transformations we applied to  $\gamma$  to obtain  $\eta$  can be now reversed, then the free homotopy class of this term is

$$g_0 t^{\varepsilon_1} g_1 \cdots g_{i-1} y t^{\varepsilon_i} g_i \cdots g_{n-1} t^{\varepsilon_n}.$$

Assume now that  $\varepsilon_i = -1$ . The term of the bracket corresponding to this intersection point is obtained by inserting  $\lambda$  right after  $\tau_i$ . This yields the element

$$g_0 t^{\varepsilon_1} g_1 \cdots g_{i-1} t^{\varepsilon_i} y g_i \cdots g_{n-1} t^{\varepsilon_n}.$$

By using the relation  $t^{-1}y = \varphi(y)t^{-1}$  we see that this element can be written as

$$g_0 t^{\varepsilon_1} g_1 \cdots g_{i-1} \varphi(y) t^{\varepsilon_i} g_i \cdots g_{n-1} t^{\varepsilon_n}.$$

To conclude, observe that pairs of terms corresponding to  $\varepsilon_i = 1$  and  $\varepsilon_i = -1$  have opposite signs because the tangents of  $\eta$  at the corresponding points point in opposite directions, and the tangent of  $\lambda$  is the same for both terms. (see Figure 7)

■

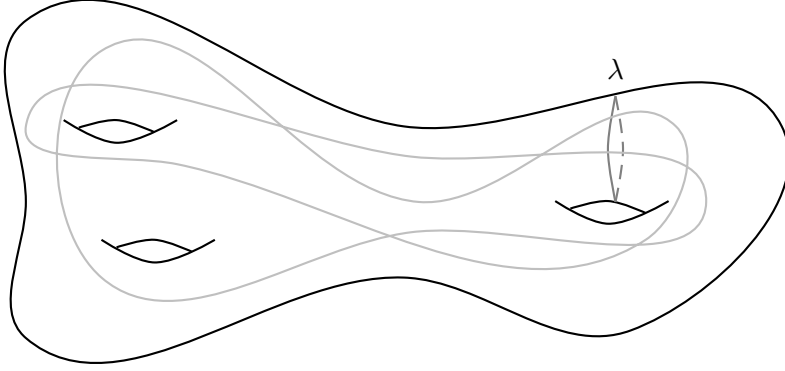


Figure 8: The intersection of a non-separating curve  $\lambda$  and another curve

## 6 Some results on surface groups

This section contains auxiliary results showing that certain equations do not hold in the fundamental group of the surface. These results will be used in Sections 7 and 8 to prove that certain sequences satisfy the hypothesis of Theorems 2.13, 2.15, 4.16 and 4.19.

We will make use of the following well known result, (see [25, Proposition 2.16].)

**Proposition 6.1.** *If  $G$  is a free group or a free abelian group and  $g$  is an element of  $G$  such that  $g^n = 1$  for some non zero integer  $n$  then  $g$  is the identity.*

Let  $F$  be a free group. An element of  $F$  is said to be *primitive* if it is a member of some basis of  $F$ .

**Proposition 6.2.** *Let  $F$  be a free group and let  $a$  be a primitive element of  $F$  and let  $A$  denote the cyclic group generated by  $a$ . Then  $A$  is malnormal in  $F$ .*

*Proof.* If  $A$  is not malnormal, then there exists an element  $g$  in  $F \setminus A$  and two non-zero integers  $n$  and  $m$  such that  $ga^m g^{-1} a^n = 1$ . Denote by  $\eta$  the map from  $F$  to the abelianization of  $F$ ,  $\eta: F \rightarrow F/[F, F]$ . We have that  $1 = \eta(ga^m g^{-1} a^n) = (m+n)\eta(a)$ . Since  $a$  is primitive,  $\eta(a) \neq 1$ . On the other hand,  $F/[F, F]$  is a free abelian group. Thus, by Proposition 6.1,  $m+n=0$ .

Thus we found two elements of the free group  $F$ ,  $a^m$  and  $g$ , which commute. Hence  $a^m$  and  $g$  are power of the same element  $c \in F$  (see [25], page 10, for a proof of this statement.) Let  $k$  be an integer such that  $a = c^k$ . By hypothesis,  $a$  is primitive, thus

$k \in \{1, -1\}$ . Consequently, either  $c = a$  or  $c = a^{-1}$ . This implies that  $g$  is a multiple of  $a$  contradicting the assumption that  $g \in F \setminus A$ . ■

If  $x$  is an element of the fundamental group of a surface and  $x$  can be represented by a simple closed curve parallel to a boundary component then  $x$  is primitive. Therefore, by Proposition 6.2 we have the following result.

**Corollary 6.3.** *Let  $\Sigma$  be an orientable surface with non-empty boundary and let  $p$  be a point in  $\Sigma$ . Let  $a$  be an element of  $\pi_1(\Sigma, p)$  which can be represented by a simple closed curve parallel to a boundary component of  $\Sigma$ . Then the cyclic group generated by  $a$  is malnormal in  $\pi_1(\Sigma, p)$ .*

**Proposition 6.4.** *Let  $F$  be a free group and let  $a$  and  $b$  be primitive elements of  $F$ . If  $n$  and  $m$  are non-zero integers such that  $a^m g b^n g^{-1} = 1$  then either  $a$  and  $b$  are conjugate or  $a$  and  $b^{-1}$  are conjugate.*

*Proof.* Denote by  $N$  the minimal normal subgroup of  $F$  containing  $b$ , and by  $\rho$  the quotient map  $\rho: F \rightarrow F/N$ . Then  $1 = \rho(a^m g b^n g^{-1}) = \rho(a)^m$ . Since we added the relation  $b = 1$  and  $b$  is primitive,  $F/N$  is a free or trivial group. By Proposition 6.1, 1 is the only element of  $F/N$  of finite order. Consequently,  $\rho(a) = 1$  and  $a \in N$ . Then there exists an integer  $k$  and  $h \in F$  such that  $a = h^{-1} b^k h$ . Since  $a$  is primitive,  $k \in \{-1, 1\}$ , as desired. ■

**Corollary 6.5.** *Let  $\Sigma_1$  and  $\varphi$  be as in Lemma 5.2. If  $\Sigma_1$  is not a cylinder then the HNN extension of Lemma 5.2 is separated.*

*Proof.* Let  $\lambda$  and  $\lambda_1$  be as in the paragraph before Lemma 5.2. Let  $a$  and  $b$  denote elements on the fundamental group of  $\Sigma_1$  such that  $\lambda$  and  $\lambda_1$  are representatives of  $a$  and  $b$  respectively. Since  $\Sigma_1$  is not the cylinder,  $a$  and  $b$  are not in the same conjugacy class. Thus,  $a$  and  $b$  are not conjugated. By Proposition 6.4, if  $m$  and  $n$  are integers then the equation  $a^m g b^n g^{-1} = 1$  does not hold. In other words, the HNN extension is separated. ■

**Proposition 6.6.** *Let  $F$  be a free group and let  $a$  and  $g$  be elements of  $F$  such that  $a$  is primitive. If  $A$  denotes the cyclic group generated by  $a$  then for every  $g \in F \setminus A$ , the elements  $g$  and  $g^{-1}$  do not belong to the same double coset mod  $A$ .*

*Proof.* Let  $n$  and  $m$  be two non zero integers. We will show that  $g a^m g a^n \neq 1$ . Let  $u = g a^m$ . Since  $g a^m g a^m a^{-m} a^n = u^2 a^{n-m}$ . Thus  $u^2 = a^{m-n}$ . Since  $a$  is primitive,  $m - n$  is even and  $u = a^{\frac{m-n}{2}}$ . Hence,  $g$  is a power of  $a$ . ■

The following result follows straightforwardly from Proposition 6.6.

**Corollary 6.7.** *Let  $\Sigma$  be an orientable surface with non-empty boundary and let  $p$  be a point in  $\Sigma$ . Let  $a$  be an element of  $\pi_1(\Sigma, p)$  such that  $a$  can be represented by a simple closed curve freely homotopic to a boundary component of  $\Sigma$ . Let  $A$  denote the cyclic group generated by  $a$ . For every  $g \in \pi_1(\Sigma, p) \setminus A$ ,  $g$  and  $g^{-1}$  do not belong to the same double coset mod  $A$ .*

**Proposition 6.8.** *Let  $\Sigma$  be an orientable surface with non-empty boundary which is not the cylinder. Let  $p$  be a point in  $\Sigma$ . Let  $a, b$  and  $g$  be elements of  $\pi_1(\Sigma, p)$  such that  $a$  and  $b$  can be represented by simple closed curves freely homotopic to distinct boundary components of  $\Sigma$ . If  $g$  is not a multiple of  $a$  nor a multiple of  $b$  then for every pair of integers  $n$  and  $m$ ,  $ga^mgb^n \neq 1$ .*

*Proof.* Assume that  $ga^mgb^n = 1$ . Notice that  $m \neq 0$  and  $n \neq 0$ . Let  $u = ga^m$ . Since  $ga^mgb^n = ga^mga^ma^{-m}b^n = u^2a^{-m}b^n$ ,

$$u^2 = b^{-n}a^m. \quad (6.1)$$

Suppose that  $\Sigma$  has exactly two boundary components. Denote by  $h$  the genus of  $\Sigma$ . Since  $\Sigma$  is not the cylinder,  $h > 1$ . Then there exists a presentation of the fundamental group of  $\Sigma$  such that the free generators are  $a, a_1, a_2, \dots, a_h, b_1, b_2, \dots, b_h$  and

$$b = aa_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1} \dots a_hb_ha_h^{-1}b_h^{-1} \quad (6.2)$$

Combining Equations (6.1) and (6.2) we obtain

$$u^2 = (aa_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1} \dots a_hb_ha_h^{-1}b_h^{-1})^{-n}a^m.$$

Observe that all the elements of the right hand side of the above equation are in the free generating set of the group. We can check that both assumptions  $n > 0$  and  $n < 0$  lead to a contradiction. Since  $n \neq 0$  the result is proved in this case.

Now, assume that  $\Sigma$  has three or more boundary components. Then there exists a free basis of the fundamental group of  $\Sigma$  containing  $a$  and  $b$ . In this case, an element of the form  $a^{-m}b^n$  cannot be equal to an element of the form  $u^2$  unless  $m = 0$  or  $n = 0$ . This concludes the proof.  $\blacksquare$

## 7 Goldman Lie algebras of oriented curves

In this section we combine some of our previous results to prove Theorem 7.7.

**Definition 7.1.** Let  $x$  and  $y$  be conjugacy classes of  $\pi_1(\Sigma, p)$  such that  $x$  can be represented by a simple loop. We associate a non-negative integer  $t(x, y)$  to  $x$  and  $y$ , called the *number of terms of  $y$  with respect to  $x$*  in the following way:

Firstly, assume that  $x$  has a separating representative. Let  $(w_1, w_2, \dots, w_n)$  be cyclically reduced sequence for the amalgamated product of Remark 3.1 such that the product  $w_1 w_2 \cdots w_n$  is conjugate to  $y$ . (The existence of such a sequence is guaranteed by Theorem 2.7.) We define  $t(x, y) = 0$  if  $n \leq 1$  and  $t(x, y) = n$  otherwise. (By Theorem 2.7,  $t(x, y)$  is well defined if  $x$  has a separating representative.)

Secondly, assume that  $x$  can be represented as a non-separating closed curve. Let  $(g_0, t^{\varepsilon_1}, g_1 \dots, g_{n-1}, t^{\varepsilon_n})$  be a cyclically reduced sequence for the HNN extension defined in Lemma 5.2 such that the product of this sequence is conjugate to  $y$ . (The existence of such a sequence is guaranteed by Theorem 4.5.) We set  $t(x, y) = n$ . (By Theorem 4.5,  $t(x, y)$  is well defined in this case.)  $\square$

Let  $\alpha$  and  $\beta$  be two curves that intersect transversally. The *geometric intersection number of  $\alpha$  and  $\beta$*  is the number of times that  $\alpha$  crosses  $\beta$ . More precisely, the geometric intersection number of  $\alpha$  and  $\beta$  is the number of pair of points  $(u, v)$ , where  $u$  is in the domain of  $\alpha$ ,  $v$  is in the domain of  $\beta$ ,  $u$  and  $v$  have the same image in  $\Sigma$  and the branch through  $u$  is transversal to the branch through  $v$  in the surface. Thus, the geometric intersection number of  $\alpha$  and  $\beta$  is the number of intersection points of  $\alpha$  and  $\beta$  counted with multiplicity.

Let  $a$  and  $b$  denote two free homotopy classes of curves. The *minimal intersection number of  $a$  and  $b$*  denoted by  $i(a, b)$  is the minimal possible geometric intersection number of pairs of curves representing  $a$  and  $b$ .

**Lemma 7.2.** Let  $x$  and  $y$  be conjugacy classes of  $\pi_1(\Sigma, p)$ . Assume that  $x$  can be represented as a simple closed curve. Then the minimal intersection number of  $x$  and  $y$  is less than or equal to the number of terms of  $y$  with respect to  $x$ . In symbols,  $i(x, y) \leq t(x, y)$ .

*Proof.* If  $x$  can be represented by a separating (resp. non-separating) curve, by Lemma 3.2 (resp. Theorem 5.3) there exist representatives of  $x$  and  $y$  with exactly  $t(x, y)$  intersection points.  $\blacksquare$

**Remark 7.3.** If  $x$  and  $y$  are conjugacy classes of  $\pi_1(\Sigma, p)$  and  $x$  can be represented as a simple closed curve  $x$ , it can be proved directly that  $i(x, y) = t(x, y)$ . Since this equality follows from Theorem 7.7, we do not give a proof of this statement here.  $\square$

**Definition 7.4.** Let  $u$  and  $v$  be conjugacy classes of  $\pi_1(\Sigma, p)$ . The *number of terms of the Goldman Lie bracket  $[u, v]$*  denoted by  $g(u, v)$  is the sum of the absolute values of



the coefficients of the expression of  $[u, v]$  in the basis of the vector space given by the set of conjugacy classes.  $\square$

**Remark 7.5.** Let  $u$  and  $v$  be conjugacy classes of  $\pi_1(\Sigma, p)$ . Since one can compute the Lie bracket by taking representatives of  $u$  and  $v$  with minimal intersection, and the bracket may have cancellation then the number of terms of the Goldman Lie bracket  $[u, v]$  is smaller or equal than the minimal intersection number of  $u$  and  $v$ . In symbols,  $g(u, v) \leq i(u, v)$ .  $\square$

Our next result states that in the closed torus, the Goldman bracket always “counts” the intersection number of two free homotopy classes.

**Lemma 7.6.** *Let  $a$  and  $b$  denote free homotopy classes of the fundamental group of the closed oriented torus  $\mathbb{T}$ . Then  $i(a, b) = g(a, b)$ . Moreover,  $[a, b] = \pm i(a, b) a \cdot b$ , where  $a \cdot b$  denotes the based loop product of a representative of  $a$  with a representative of  $b$ .*

*Proof.* Assume first that  $a$  and  $b$  are not a proper powers. Thus  $a$  admits a simple representative  $\alpha$ , and there exist a class  $c$  which admits a simple representative  $\delta$  and such that intersects  $\alpha$  exactly once with intersection number  $+1$ . Whence,  $a$  and  $c$  are a basis of fundamental group of  $\mathbb{T}$ .

Let  $k$  and  $l$  be integers such that  $b = a^k c^l$  (Recall that the fundamental group of the closed torus is abelian). The universal cover of  $\mathbb{T}$  is the euclidean plane  $\mathbb{R}^2$ . We can consider a projection map  $p: \mathbb{R}^2 \rightarrow \mathbb{T}$  such that the liftings of  $\alpha$  are the horizontal lines of equation  $y = n$  with  $n \in \mathbb{Z}$  and the liftings of  $\delta$  are vertical lines with equation  $x = n$  with  $n \in \mathbb{Z}$ . Thus there exist a representative  $\beta$  of  $b$  such that the liftings of  $\beta$  are lines of equation  $y = \frac{l}{k}x + n$  with  $n \in \mathbb{Z}$ .

One can check that the intersection of  $\alpha$  and  $\delta$  is exactly the projection of  $\{(0, 0), (\frac{k}{l}, 0), (2\frac{k}{l}, 0), \dots, ((l-1)\frac{k}{l}, 0)\}$ . Moreover, all these points project to distinct points and the sign of the intersection of  $\alpha$  and  $\delta$  at each of these points is equal to the sign of  $l$ .

Thus  $[a, b] = [a, a^k c^l] = l a^{k+1} c^l$ . Thus  $g(a, b) = |l|$ . Since there are representatives of  $a$  and  $b$  intersecting in  $l$  points,  $i(a, b) \leq g(a, b)$ . By Remark 7.5,  $i(a, b) = g(a, b)$ .

Assume now  $a$  and  $b$  are classes which are proper powers. Thus there exist simple closed curves  $\epsilon$  and  $\gamma$  and integers  $i$  and  $j$  such that  $\epsilon^i$  is a representative of  $a$  and  $\gamma^j$  is a representative of  $b$ . Moreover by the first part of this proof, we can assume that if  $e$  denotes the free homotopy class of  $\epsilon$  and  $g$  denotes the free homotopy class of  $\gamma$ , then  $\epsilon$  and  $\gamma$  intersect in exactly  $i(e, g)$  points. Take  $i$  “parallel” copies of  $\epsilon$  very close to each other and reconnect them so that they form a representative  $\alpha$  of  $a$ . Do the same with  $j$  copies of  $\gamma$ , and denote by  $\beta$  the representative of  $b$  we obtain. We can perform this

surgery far from the intersection points of  $\epsilon$  and  $\gamma$  so that the number of intersection points of  $\alpha$  and  $\beta$  is exactly  $i \cdot j \cdot i(\epsilon, \gamma)$ , whence  $(i \cdot j \cdot i(e, g)) \leq i(a, b)$ . On the other hand, the bracket is  $[a, b] = \pm(i \cdot j \cdot i(e, g)) (a \cdot b)$ . By Remark 7.5,  $g(a, b) = i(a, b)$ , as desired. ■

Here is a sketch of the proof of our next result (the detailed proof follows the statement): Given a free homotopy class  $x$  with a simple representative and an arbitrary class  $y$ , we can write the bracket  $[x, y]$  as in Theorem 3.4 or as in Theorem 5.3. The conjugacy classes of the terms of this bracket have representatives as the ones studied in Theorem 2.13 or in Theorem 4.16. By Corollary 6.3 (resp. Corollary 6.7) the free product on amalgamation (resp. the HNN extension) we are considering satisfies the hypothesis of Theorem 2.13 (resp. Theorem 4.16.) This implies that the terms of the bracket do not cancel.

**Theorem 7.7.** *Let  $x$  be a free homotopy class that can be represented by a simple closed curve on  $\Sigma$  and let  $y$  be any free homotopy class. Then the following non-negative integers are equal.*

- (1) *The minimal number of intersection points of  $x$  and  $y$ .*
- (2) *The number of terms in the Goldman Lie bracket  $[x, y]$ , counted with multiplicity.*
- (3) *The number of terms in the reduced sequence of  $y$  determined by  $x$ .*

In symbols,  $i(x, y) = g(x, y) = t(x, y)$ .

*Proof.* By Remarks 7.2 and 7.5,  $g(x, y) \leq i(x, y) \leq t(x, y)$ . Hence it is enough to prove that  $t(x, y) \leq g(x, y)$ . By Lemma 7.6, we can assume that  $\Sigma$  is not the torus.

We first prove that  $t(x, y) \leq g(x, y)$  when  $x$  can be represented by a separating simple closed curve  $\chi$ . By Theorem 2.7, there exists a cyclically reduced sequence  $(w_1, w_2, \dots, w_n)$  for the free product of amalgamation determined by  $\chi$  in Remark 3.1 such that the product  $w_1 w_2 \cdots w_n$  is a representative of  $y$ . If  $n = 0$  or  $n = 1$ , then  $t(x, y) = 0$  and the result holds. If  $n > 1$ , by Theorem 3.4, there exists  $s \in \{1, -1\}$  such that

$$s [x, y] = \sum_{i=1}^n (-1)^i w_1 w_2 \cdots w_i x w_{i+1} \cdots w_n. \quad (7.1)$$

If  $i$  and  $j$  are such that there is cancellation between the  $i$ -th term and the  $j$ -th term of the right hand side of the Equation (7.1), then  $(-1)^i = -(-1)^j$ . Consequently,  $i$  and  $j$  have different parity.

We will work at the basepoint indicated above and will abuse notation by pretending  $x$  and  $y$  are elements of the fundamental group of  $\Sigma$  (see Notation 3.3.) By Corollary 6.3,

the cyclic group generated by  $x$  is malnormal in  $\pi_1(\Sigma_1, p)$  and is malnormal in  $\pi_1(\Sigma_2, p)$ , where  $\Sigma_1$  and  $\Sigma_2$  are as in Remark 3.1. Thus, the hypothesis of Theorem 2.13 hold for this free product with amalgamation. Hence, by Theorem 2.13, there is no cancellation in the sum of the right hand side of Equation (7.1). Consequently,  $t(x, y) = g(x, y)$ .

Now we prove the result when  $x$  can be represented by a non-separating simple closed curve  $\lambda$ . Consider the HNN extension of Lemma 5.2 determined by  $\lambda$ . By Theorem 4.5 there exists a cyclically reduced sequence  $(g_0, t^{\varepsilon_1}, g_1, t^{\varepsilon_2}, \dots, g_{n-1}, t^{\varepsilon_n})$  such that the product  $g_0 t^{\varepsilon_1} g_1 t^{\varepsilon_2} \dots g_{n-1} t^{\varepsilon_n}$  is a representative of  $y$ . By Theorem 5.3, there exists  $s \in \{1, -1\}$  such that  $s[x, y]$  equals

$$\sum_{i : \varepsilon_i = 1} g_0 t^{\varepsilon_1} g_1 t^{\varepsilon_2} \dots g_{i-1} x t^{\varepsilon_i} g_i \dots g_{n-1} t^{\varepsilon_n} - \sum_{i : \varepsilon_i = -1} g_0 t^{\varepsilon_1} g_1 t^{\varepsilon_2} \dots g_{i-1} \varphi(x) t^{\varepsilon_i} \dots g_{n-1} t^{\varepsilon_n}.$$

If  $t(x, y) > g(x, y)$  then there is cancellation in the above sum. Therefore, there exist two integers  $h$  and  $k$  such that  $\varepsilon_h = 1$ ,  $\varepsilon_k = -1$  and the products

$$g_0 t^{\varepsilon_1} g_1 t^{\varepsilon_2} \dots g_{h-1} x t^{\varepsilon_h} t^{\varepsilon_{h+1}} \dots g_{n-1} t^{\varepsilon_n} \text{ and } g_0 t^{\varepsilon_1} g_1 t^{\varepsilon_2} \dots g_{k-1} \varphi(x) t^{\varepsilon_k} \dots g_{n-1} t^{\varepsilon_n}$$

are conjugate. We use now the notations of Lemma 5.2 and the paragraph before Lemma 5.2. By Corollary 6.3,  $\pi_1(\lambda, p)$  and  $\pi_1(\lambda_1, p)$  are malnormal in  $\pi_1(\Sigma_1, p)$ . Since we are assuming that  $\Sigma$  is not a torus, then  $\Sigma_1$  is not a cylinder. Then by Corollary 6.5 the HNN extension is separated. By Theorem 4.16 the two products above are not conjugate, a contradiction. Thus the proof is complete.  $\blacksquare$

**Corollary 7.8.** *If  $x$  and  $y$  are conjugacy classes of curves that can be represented by simple closed curves then  $t(x, y) = t(y, x)$ .*

**Remark 7.9.** Let  $x$  be a free homotopy class that can be represented by a simple closed curve on  $\Sigma$  and let  $y$  be any free homotopy class.  $\square$

## 8 Goldman Lie algebras of unoriented curves

Recall that Goldman [13] defined a Lie algebra of unoriented loops as follows. Denote by  $\pi^*$  the set of conjugacy classes of  $\pi_1(\Sigma, p)$ . For each  $x \in \pi^*$ , denote by  $\bar{x}$  the conjugacy class of a representative of  $x$  with opposite orientation. Set  $\hat{x} = x + \bar{x}$  and  $\hat{\pi} = \{\bar{x} + x, x \in \pi^*\}$ . The map  $\hat{\cdot}$  is extended linearly to the vector space of linear combinations of elements of  $\pi^*$ . Denote by  $V$  the real vector space generated by  $\hat{\pi}$ , that is, the image of the map  $\hat{\cdot}$ . For each pair of elements of  $\pi^*$ ,  $x$  and  $y$ , define the unoriented bracket

$$[\hat{x}, \hat{y}] = ([x, y] + [\bar{x}, \bar{y}]) + ([x, \bar{y}] + [\bar{x}, y]) = \widehat{[x, y]} + \widehat{[x, \bar{y}]}.$$

We denote the bracket of oriented curves and the bracket of unoriented curves by the same symbol,  $[\ , \ ]$ .

An *unoriented term* of the bracket  $[a + \bar{a}, b + \bar{b}] = [\hat{a}, \hat{b}]$  of a pair of unoriented curves  $\hat{a} = a + \bar{a}$  and  $\hat{b} = b + \bar{b}$  is a term of the form  $c(z + \bar{z})$ , where  $c$  is an integer and  $z$  is a conjugacy class, that is, an elements of the basis of  $V$  multiplied by an integer coefficient.

Let  $u(x, y)$  denote the number of unoriented terms of the bracket  $[\hat{x}, \hat{y}]$  considered as a bilinear map on  $V$ , counted with multiplicity. This is the sum of the absolute value of the coefficients of the expression of  $[\hat{x}, \hat{y}]$  in the basis  $\hat{\pi}$ .

**Example 8.1.** With the notations of Lemma 7.6,

$$[\hat{a}, \hat{b}] = [\hat{a}, \widehat{a^k c^l}] = \pm i(a, b)(\widehat{a^{k+1} c^l} - \widehat{a^{k-1} c^l}).$$

Using the fact that every free homotopy class of curves in the torus admits a multiple of a simple closed curve as representative, we can show that for every pair of free homotopy classes  $x$  and  $y$ ,

$$u(x, y) = 2 \cdot i(x, y) = 2 \cdot g(x, y) = 2 \cdot t(x, y).$$

□

The strategy of the proof of our next theorem is similar to that of Theorem 7.7, namely, we write the terms of the bracket in a certain form (using Theorem 3.4 or Theorem 5.3). By the results on Section 6, we can apply Theorems 2.15 and 4.19 to show that the pairs of conjugacy classes of these sums which have different signs are distinct by Theorems 2.15 and 4.19.

**Theorem 8.2.** *Let  $x$  and  $y$  be conjugacy classes of  $\pi_1(\Sigma, p)$  such that  $x$  can be represented by a simple closed curve. Then the following non-negative integers are equal*

- (1) *The number of unoriented terms of the bracket  $[\hat{x}, \hat{y}]$ ,  $u(x, y)$ .*
- (2) *Twice the minimal number of intersection points of  $x$  and  $y$ ,  $2 \cdot i(x, y)$ .*
- (3) *Twice the number of terms of the Goldman bracket on oriented curves, counted with multiplicity,  $2 \cdot g(x, y)$ .*
- (4) *Twice the number of terms of the sequence of  $y$  with respect to  $x$ ,  $2 \cdot t(x, y)$ .*

*In symbols,*

$$u(x, y) = 2 \cdot i(x, y) = 2 \cdot g(x, y) = 2 \cdot t(x, y).$$

*Proof.* By Remark 7.5,  $u(x, y) \leq 2 \cdot i(x, y) = 2 \cdot t(x, y)$ . By Example 8.1, we can assume that  $\Sigma$  is not the torus. By Theorem 7.7, it is enough to prove that  $u(x, y) = 2 \cdot t(x, y)$ . Assume that  $u(x, y) < 2 \cdot t(x, y)$ .

By definition the bracket  $[\widehat{x}, \widehat{y}]$  is an algebraic sum of terms of the form  $\widehat{z} = z + \overline{z}$ , where  $z$  is a conjugacy class of curves and  $z$  and  $\overline{z}$  are terms of one of the four following brackets:  $[x, y]$ ,  $[x, \overline{y}]$ ,  $[\overline{x}, y]$  and  $[\overline{x}, \overline{y}]$ . By Theorem 7.7, the number of terms of each of the four brackets above is  $t(x, y)$ . Since  $u(x, y) < 2 \cdot t(x, y)$  there has to be one term belonging to one of above four brackets that cancels with a term of another of those brackets. Denote one of these terms that cancel by  $t_1$  and the other by  $t_2$ . By Theorem 7.7, there is no inner cancellation in any of the above four brackets. Consequently, if  $t_1$  is a term of  $[u, v]$ , where  $u \in \{x, \overline{x}\}$  and  $v \in \{y, \overline{y}\}$  then  $t_2$  is a term of one of the following brackets:  $[u, \overline{v}]$ ,  $[\overline{u}, v]$ , or  $[\overline{u}, \overline{v}]$ . Hence it suffices to assume that  $t_1$  is a term of  $[x, y]$  and to analyze each of the following three cases.

- (1)  $t_2$  is a term of  $[\overline{x}, y]$
- (2)  $t_2$  is a term of  $[x, \overline{y}]$
- (3)  $t_2$  is a term of  $[\overline{x}, \overline{y}]$

Assume first that  $x$  can be represented by a separating curve  $\chi$ . By Theorem 2.7, there exists a cyclically reduced sequence  $(w_1, w_2, \dots, w_n)$  in the amalgamating product of Remark 3.1 determined by  $\chi$  such that the product  $w_1 w_2 \cdots w_n$  is a representative of  $y$ . By Theorem 3.4, then there exists  $s \in \{1, -1\}$  and  $i, j \in \{1, 2, \dots, n\}$  such that  $t_1 = s(-1)^i w_1 w_2 \cdots w_i x w_{i+1} \cdots w_n$  and one of the following holds.

- (1)  $t_2 = s(-1)^{j+1} w_1 w_2 \cdots w_j x^{-1} w_{j+1} \cdots w_n$ .
- (2)  $t_2 = s(-1)^{j+1} w_n^{-1} w_{n-1}^{-1} \cdots w_{j+1}^{-1} x w_j^{-1} \cdots w_1^{-1}$ .
- (3)  $t_2 = s(-1)^j w_n^{-1} w_{n-1}^{-1} \cdots w_{j+1}^{-1} x^{-1} w_j^{-1} \cdots w_1^{-1}$ .

(Note that when we change direction of one of the elements of the bracket,  $x$  or  $y$ , there is a factor  $(-1)$  because one of the tangent vectors at the intersection point has an opposite direction. Also, changing direction of  $x$  and  $y$  does not change signs.)

Let us study first case (1). Clearly, if  $t_1$  and  $t_2$  cancel then  $(-1)^i = -(-1)^{j+1}$  and the products  $w_1 w_2 \cdots w_i x w_{i+1} \cdots w_n$  and  $w_1 w_2 \cdots w_j x^{-1} w_{j+1} \cdots w_n$  are conjugate. Therefore,  $i$  and  $j$  have equal parities. By Corollary 6.3, the subgroup generated by  $x$  is malnormal in the two amalgamated groups of the amalgamated product of Remark 3.1. (We are again treating  $x$  as an element of the fundamental group of the surface.) Hence we can apply Theorem 2.13, with  $a = x$  and  $b = \overline{x}$  to show that  $w_1 w_2 \cdots w_i x w_{i+1} \cdots w_n$  and  $w_1 w_2 \cdots w_j x^{-1} w_{j+1} \cdots w_n$  are not conjugate, a contradiction.

Similarly, by Theorem 2.15 and Corollary 6.7, Cases (2) and (3) are not possible.

Now assume that  $x$  has a non-separating representative. By Theorem 4.5 there exist a cyclically reduced sequence  $(g_0, t^{\varepsilon_1}, g_1, \dots, g_{n-1}, t^{\varepsilon_n})$  whose product is an element of  $y$ . By Theorem 5.3 there exist an integer  $i$  such that the term  $t_1$  has the form

$s\varepsilon_i \cdot g_0 t^{\varepsilon_1} g_1 \cdots g_i u t^{\varepsilon_i} \cdots g_{n-1} t^{\varepsilon_n}$  where  $u = x$  if  $\varepsilon_i = 1$  and  $u = \varphi(x)$  if  $\varepsilon_i = -1$ . By Theorem 5.3 there exist an integer  $j$  such that the term  $t_2$  has one of the following forms.

- (1)  $t_2 = -s\varepsilon_j \cdot g_0 t^{\varepsilon_1} g_1 \cdots g_j v t^{\varepsilon_j} \cdots g_{n-1} t^{\varepsilon_n}$  where  $v = x^{-1}$  if  $\varepsilon_j = 1$  and  $v = \varphi(x^{-1})$  if  $\varepsilon_j = -1$ .
- (2)  $t_2 = s\varepsilon_j \cdot g_{n-1}^{-1} t^{-\varepsilon_{n-1}} g_{n-2}^{-1} t^{-\varepsilon_{n-2}} \cdots g_j^{-1} v t^{-\varepsilon_j} \cdots g_1^{-1} t^{-\varepsilon_1} g_0^{-1} t^{-\varepsilon_n}$ , where  $v = x$  if  $\varepsilon_j = -1$  and  $v = \varphi(x)$  if  $\varepsilon_j = 1$ .
- (3)  $t_2 = -s\varepsilon_j \cdot g_{n-1}^{-1} t^{-\varepsilon_{n-1}} g_{n-2}^{-1} t^{-\varepsilon_{n-2}} \cdots g_j^{-1} v^{-1} t^{-\varepsilon_j} \cdots g_1^{-1} t^{-\varepsilon_1} g_0^{-1} t^{-\varepsilon_n}$  where  $v = x$  if  $\varepsilon_j = -1$  and  $v = \varphi(x)$  if  $\varepsilon_j = 1$ .

The argument continues similarly to that of the separating case: By Corollary 6.3 and Corollary 6.5 the HNN is separated and the subgroups we are considering are malnormal. By Theorem 4.16,  $t_2$  cannot have the form described in Case (1). (We need to use the fact that a non-trivial element and its inverse are not conjugate in an infinite cyclic group). Cases (2) and (3) are ruled out by Theorem 4.19 and Corollary 6.7 and Proposition 6.8. ■

Let  $n$  be a positive integer and let  $x$  be a free homotopy class with representative  $\chi$ . Denote by  $x^n$  the conjugacy class of the curve that wraps  $n$  times around  $\chi$ . We can extend Theorem 8.2 to the case of multiple curves using the same type of arguments.

**Theorem 8.3.** *Let  $n$  be a positive integer and let  $x$  and  $y$  be conjugacy classes of  $\pi_1(\Sigma, p)$  such that  $x$  can be represented by a simple closed curve  $\chi$ . Then the following equalities hold.*

$$u(x^n, y) = n \cdot u(x, y) = 2i(x^n, y) = 2 \cdot n \cdot i(x, y) = 2 \cdot g(x^n, y) = 2 \cdot n \cdot g(x, y) = 2 \cdot n \cdot t(x, y).$$

The next lemma is well known but we did not find an explicit proof in the literature. A stronger version of this result (namely,  $i(x, x)$  equals twice the minimal number of self-intersection points of  $x$ ) is proven in [9] for the case of surfaces with boundary.

**Lemma 8.4.** *If  $x$  is a homotopy class which does not admit simple representatives then the minimal intersection number of  $x$  and  $x$  is not zero. In symbols,  $i(x, x) \neq 0$ .*

*Proof.* Let  $\alpha$  and  $\beta$  be two transversal representatives of  $x$ . Let  $\chi$  be a representative of  $x$  with minimal number of self-intersection points and let  $P$  be a self-intersection point of  $\chi$ . Let  $p: H \rightarrow \Sigma$  be the universal cover of  $\Sigma$ . Consider two distinct lifts of  $\chi$ ,  $\tilde{\chi}_1$  and  $\tilde{\chi}_2$  which intersect at a point  $Q$  of  $H$  such that  $p(Q) = P$ . Consider a lift of  $\alpha$ ,  $\tilde{\alpha}$  such that the endpoints of  $\tilde{\alpha}$  and  $\tilde{\chi}_1$  coincide. Analogously, consider a lift of  $\beta$ ,  $\tilde{\beta}$  such that the endpoints of  $\tilde{\beta}$  and  $\tilde{\chi}_2$  coincide. Since  $\chi$  intersects in a minimal number

of points, the endpoints of the lifts  $\chi_1$  and  $\chi_2$  are linked. Thus, the endpoints of  $\alpha_1$  and  $\tilde{\beta}$  are linked. Hence,  $\tilde{\alpha}$  and  $\tilde{\beta}$  intersect in  $H$ . Consequently,  $\alpha$  and  $\beta$  intersect in  $\Sigma$ . ■

Our next result is a global characterization of free homotopy classes with simple representatives in terms of the Goldman Lie bracket.

**Corollary 8.5.** *Let  $x$  be a free homotopy class. Then  $x$  has a multiple of a simple representative if and only if for every free homotopy class  $y$  the number of terms of the bracket of  $x$  and  $y$  is equal to the minimal intersection number of  $x$  and  $y$ . In symbols,  $x$  has a multiple of a simple representative if and only if  $g(x, y) = i(x, y)$  for every free homotopy class  $y$ .*

*Proof.* If  $x$  has a simple representative and  $y$  is an arbitrary free homotopy class then  $g(x, y) = i(x, y)$  by Theorem 7.7. If  $x$  does not have a simple representative then by Lemma 8.4,  $i(x, x) \geq 1$ . On the other hand, by the antisymmetry of the Goldman Lie bracket we have  $[x, x] = 0$ . Thus,  $g(x, x) = 0$  and the proof of the corollary is complete. ■

## 9 Examples

The assumption in Theorems 7.7 and 8.2 that one of the curves is simple cannot be dropped. Goldman [13] gave the following example, (attributed to Peter Scott): For any conjugacy class  $a$ , the Lie bracket  $[a, a] = 0$ . On the other hand, if  $a$  cannot be represented by a multiple of a simple curve, then any two representatives of  $a$  cannot be disjoint.

Here is a family of examples:

**Example 9.1.** Consider the conjugacy classes of the curves  $aab$  and  $ab$  in the pair of pants (see Figure 9.) The term of the bracket corresponding to  $p_1$  is the conjugacy class of  $aabba$  and the term of the bracket corresponding to  $p_2$  is  $baaab$ . The conjugacy classes of both terms are the same, and the signs are opposite. Then the Goldman bracket of these conjugacy classes is zero. Nevertheless, the minimal intersection number is two (see Figure 9.)

More generally, for every pair of positive integers  $n$  and  $m$ , the curves  $a^n b$  and  $a^m b$  have minimal intersection  $2 \min(m, n)$ . Nevertheless, the bracket of these pairs is zero. In symbols,  $[a^n b, a^m b] = 0$ . (The intersection number as well as the Goldman Lie bracket can be computed using results in [3].) □

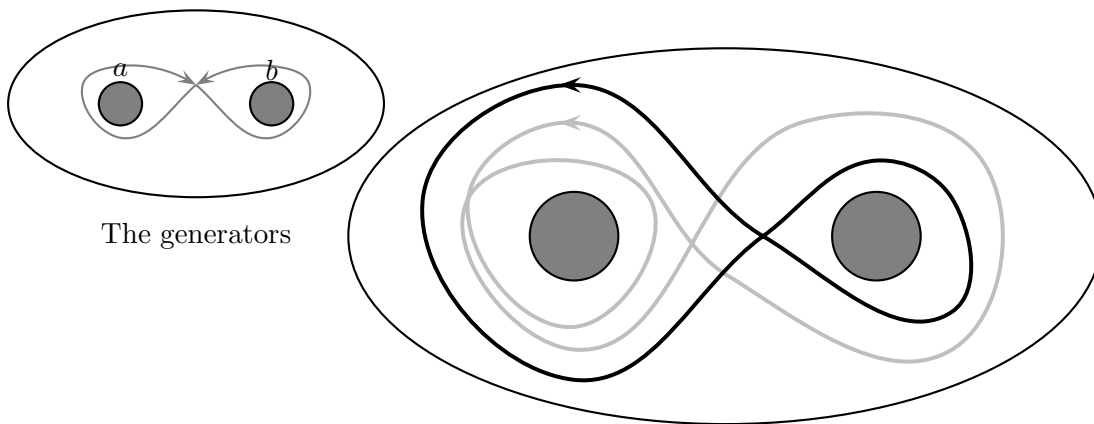


Figure 9: An example of a pair of non-disjoint curves with bracket 0

## 10 Application: Factorization of Thurston's map

Denote by  $C(\Sigma)$  the set of all conjugacy classes of curves on a surface  $\Sigma$  which admit a simple representative. Denote by  $W$  the vector space with basis all conjugacy classes of the fundamental group of  $\Sigma$ . Consider the map  $\phi: C(\Sigma) \longrightarrow W^{C(\Sigma)}$ , defined by  $\phi(a)(b) = [a, b]$ . For each (reduced) linear combination  $c$  of elements of the vector space  $W$ , define a map  $\text{abs}: W \longrightarrow \mathbb{Z}_{\geq 0}$ , where  $\text{abs}(c)$  is the sum of the absolute value of the coefficients of  $c$ .

By Theorem 7.7 the composition  $\text{abs} \circ \phi: C(\Sigma) \longrightarrow \mathbb{Z}_{\geq 0}^{C(\Sigma)}$  is the map defined by Thurston in [29] which is used to define his mapping class group invariant compactification of Teichmüller space.

## 11 Application: Decompositions of the vector space generated by conjugacy classes

For each  $w$  in  $W$ , the vector space of free homotopy classes of curves, the *adjoint map determined by  $w$* , denoted by  $\text{ad}_w$  is defined for each  $y \in W$  by  $\text{ad}_w(y) = [y, w]$ .

Let  $a$  denote the conjugacy class of a closed curve on a surface  $\Sigma$  and let  $n$  be a non-negative integer. Denote by  $W_n(a)$  the subspace of  $W$  generated by the set of conjugacy classes of oriented curves with minimal number of intersection points with  $a$  equal to  $n$ .

In this section we prove that if  $a$  is the conjugacy class of a simple closed curve then



$W_n(a)$  is invariant under  $\text{ad}_a$ . Moreover, we give a further decomposition of  $W_n(a)$  into subspaces invariant under  $\text{ad}_a$ .

### 11.1 The separating case

Let  $\chi$  be a separating simple closed curve. Let  $G *_C H$  be the amalgamated free product defined by  $\chi$  in Remark 3.1. Let  $(w_1, w_2, \dots, w_n)$  be a cyclically reduced sequence for this amalgamated free product. Denote by  $W(w_1, w_2, \dots, w_n)$  the subspace generated by the conjugacy classes which have representatives of the form  $v_1 v_2 \cdots v_n$  where for each  $i$  in  $\{1, 2, \dots, n\}$ ,  $v_i$  is an element of the double coset  $Cw_i C$ .

The following result is an immediate consequence of Theorem 3.4 and the definition of the subspaces  $W(w_1, w_2, \dots, w_n)$ .

**Proposition 11.1.** *With the notations above, the subspace  $W(w_1, w_2, \dots, w_n)$  is invariant under  $\text{ad}_x$ , the adjoint map determined by  $x$ .*

**Proposition 11.2.** *With the above notations, the subspace  $W_n(x)$  is the disjoint union of  $W(w_1, w_2, \dots, w_n)$ , where  $(w_1, w_2, \dots, w_n)$  runs over all cyclically reduced sequences of  $n$  terms of the free product with amalgamation determined by  $x$ . Therefore,  $W_n(x)$  is invariant under  $\text{ad}_x$ .*

*Proof.* It is enough to prove the result for every element of the basis of  $W_n(x)$ . Let  $y \in W_n(x)$  be a conjugacy class whose minimal intersection number with  $x$  is  $n$ . By Theorem 2.7 and Theorem 7.7, there exists a cyclically reduced sequence  $(w_1, w_2, \dots, w_n)$  with  $n$  terms for the amalgamating product of Remark 3.1 determined by a representative of  $x$  with product in  $y$ .

By Theorem 3.4, the conjugacy classes associated with each of the terms of the bracket  $[y, x]$  have the form  $w_1 w_2 \cdots w_i x w_{i+1} \cdots w_n$  for some  $i \in \{1, 2, \dots, n\}$ . The sequence  $(w_1, w_2, \dots, w_i x, w_{i+1}, \dots, w_n)$  is cyclically reduced for every  $i \in \{1, 2, \dots, n\}$ . By Theorem 7.7, the minimal intersection number of the product  $w_1 w_2 \cdots w_i x w_{i+1} \cdots w_n$  and  $a$  is  $n$ . Hence each term of the Lie bracket  $[y, x]$  is in  $W_n(x)$ . ■

**Remark 11.3.** It is not hard to see that the subspaces  $W(w_1, w_2, \dots, w_n)$  are also invariant under the map induced by the Dehn twist around  $x$ . Indeed, using Lemma 3.2, it is not hard to see that the Dehn twist around  $x$  of the conjugacy class of  $w_1 w_2 \cdots w_n$  can be represented by

$$w_1 x w_2 \bar{x} w_3 x w_4 \bar{x} \cdots w_n \bar{x} \text{ or } w_1 \bar{x} w_2 x w_3 \bar{x} \cdots w_n x$$

where the choice between two conjugacy classes above is determined by the orientation of the surface, the orientation of  $x$  and the subgroup which  $w_1$  belongs to. □

**Question 11.4.** Denote by  $X$  the cyclic group of automorphisms of  $W$  generated by  $\text{ad}_x$ . Let  $(w_1, w_2, \dots, w_n)$  be a cyclically reduced sequence. It is not hard to see that the subspace  $W(w_1, w_2, \dots, w_n)$  is invariant under  $X$ . Is  $W(w_1, w_2, \dots, w_n)$  the minimal (with respect to inclusion) subspace of  $W$  containing the conjugacy class of  $w_1 w_2 \cdots w_n$  and invariant under  $X$ .  $\square$

## 11.2 The non-separating case

We now develop the analog of Subsection 11.1 for the non-separating case. In the separating case, the terms of cyclic sequences of double cosets belong to alternating subgroups. In the non-separating case, there is a further piece of information, the sequence of  $\varepsilon$ 's. Thus, the arguments, although essentially the same, have some extra technical complications. Also, there are more invariant subspaces surfacing.

Let  $\gamma$  denote a separating curve. Let  $G^{*\varphi}$  be the HNN extension constructed in Lemma 5.2. with  $\gamma$ . Let  $(g_0, t^{\varepsilon_1}, g_1, t^{\varepsilon_2}, \dots, g_{n-1}, t^{\varepsilon_n})$  denote a cyclically reduced sequence. Denote by  $W(g_0, \varepsilon_1, g_1, \varepsilon_2, \dots, g_{n-1}, \varepsilon_n)$  the subspace of  $W$  generated by the conjugacy classes which have representatives of the form  $h_0 t^{\varepsilon_1} h_1 t^{\varepsilon_2} \cdots h_{n-1} t^{\varepsilon_n}$  where for each  $i \in \{1, 2, \dots, n\}$ ,  $h_i$  is an element of the double coset  $C_{-\varepsilon_i} w_i C_{\varepsilon_{i+1}}$ . Let  $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$  be a sequence of integers such that for each  $i \in \{1, 2, \dots, n\}$ ,  $\varepsilon_i \in \{-1, 1\}$ . Denote by  $W(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$  the subspace of  $W$  generated by the conjugacy classes which have representatives of the form  $h_0 t^{\varepsilon_1} h_1 t^{\varepsilon_2} \cdots h_{n-1} t^{\varepsilon_n}$  where  $(h_0, \varepsilon_1, h_1, \varepsilon_2, \dots, h_{n-1}, \varepsilon_n)$  is a cyclically reduced sequence.

The following result is a direct consequence of Theorem 5.3 and the definition of the subspaces  $W(g_0, \varepsilon_1, g_1, \varepsilon_2, \dots, g_{n-1}, \varepsilon_n)$  and  $W(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$ .

**Proposition 11.5.** *With the above notations, the subspaces  $W(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$  and  $W(g_0, \varepsilon_1, g_1, \varepsilon_2, \dots, g_{n-1}, \varepsilon_n)$  are invariant under  $\text{ad}_y$ , the adjoint map determined by  $y$ . Moreover,  $W(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$  is a disjoint union of subspaces of the form  $W(g_0, \varepsilon_1, g_1, \varepsilon_2, \dots, g_{n-1}, \varepsilon_n)$  where  $(g_0, \varepsilon_1, g_1, \varepsilon_2, \dots, g_{n-1}, \varepsilon_n)$  runs over cyclically reduced sequences with a sequences of  $\varepsilon$ 's given by  $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$ .*

With arguments similar to those of the proof of Proposition 11.2, we can prove the following result.

**Proposition 11.6.** *With the above notations, the subspace  $W_n(y)$  is the disjoint union of  $W(g_0, \varepsilon_1, g_1, \varepsilon_2, \dots, g_{n-1}, \varepsilon_n)$  where  $W(g_0, \varepsilon_1, g_1, \varepsilon_2, \dots, g_{n-1}, \varepsilon_n)$  runs over all cyclically reduced sequences of  $n$  terms of the HNN extension determined by  $y$ . Therefore,  $W_n(y)$  is invariant under  $\text{ad}_y$ .*

**Remark 11.7.** It is not hard to see that the subspaces  $W(g_0, \varepsilon_1, g_1, \varepsilon_2, \dots, g_{n-1}, \varepsilon_n)$  are also invariant under the map induced by the Dehn twist around  $y$ . Indeed, using Lemma 3.2, one sees that the Dehn twist around  $y$  of the conjugacy class of  $g_0, t^{\varepsilon_1}, g_1, t^{\varepsilon_2}, \dots, g_{n-1}, t^{\varepsilon_n}$  (where  $(g_0, t^{\varepsilon_1}, g_1, t^{\varepsilon_2}, \dots, g_{n-1}, t^{\varepsilon_n})$  is a cyclically reduced sequence) is represented by one of the following products:

$$g_0 u_1 t^{\varepsilon_1} g_1 u_1 t^{\varepsilon_2} \dots g_{n-1} u_n t^{\varepsilon_n},$$

where one of the following holds.

- (1) For each  $i \in \{1, 2, \dots, n\}$ , if  $\varepsilon_i = 1$  then  $u_i = y$  and if  $\varepsilon_i = -1$  then  $u_i = \varphi(\overline{y})$ .
- (2) For each  $i \in \{1, 2, \dots, n\}$ , if  $\varepsilon_i = 1$  then  $u_i = \overline{y}$  and if  $\varepsilon_i = -1$  then  $u_i = \varphi(y)$ .

where the choice between (1) and (2) is determined by the orientation of the surface and the orientation of  $y$ .  $\square$

## 12 Application: The mapping class group and the curve complex

Let  $\Sigma$  be a compact oriented surface. By  $\Sigma_{g,b}$  we denote an oriented surface with genus  $g$  and  $b$  boundary components. If  $\Sigma$  is a surface, we denote by  $\text{Mod}(\Sigma)$  the *mapping class group* of  $\Sigma$ , that is set of homotopy classes of orientation preserving homeomorphisms of  $\Sigma$ . We study automorphisms of the Goldman Lie algebra that are related to the mapping class group. The first two results, Theorems 12.5 and 12.6, apply to all surfaces. The stronger result, Theorem 12.7, applies only to surfaces with boundary.

Now we recall the curve complex, defined by Harvey in [15]. The *curve complex* of  $\Sigma$  is denoted by  $C(\Sigma)$  is the simplicial complex whose vertices are isotopy classes simple closed curves on  $\Sigma$  which are not null-homotopic and not homotopic to a boundary component. If  $\Sigma \neq \Sigma_{0,4}$  and  $\Sigma \neq \Sigma_{1,1}$  then a set of  $k+1$  vertices of the curve complex is the 0- skeleton of a  $k$ -simplex if the corresponding minimal intersection number of all pairs of vertices is zero, that is, if every pair of vertices have disjoint representatives.

For  $\Sigma_{0,4}$  and  $\Sigma_{1,1}$  two vertices are connected by an edge when the curves they represent have minimal intersection (2 in the case of  $\Sigma_{0,4}$ , and 1 in the case of  $\Sigma_{1,1}$ ).

The following isomorphism is a theorem of Ivanov [17] for the case of genus at least two. Korkmaz [20] proved the result for genus at most one and Luo gave another proof that covers all possible genera. [23]. Our discussion below is based on the formulation of Margalit [27].

**Theorem 12.1.** (Ivanov-Korkmaz-Luo) Let  $\Sigma$  be an orientable surface of negative Euler characteristic and let  $h: \text{Mod}(\Sigma) \longrightarrow \text{Aut } C(\Sigma)$  be the natural map. If  $\Sigma \neq \Sigma_{0,3}$  then

- (1)  $h$  is surjective if and only if  $\Sigma \neq \Sigma_{1,2}$  and
- (2)  $h$  is injective if and only if  $\Sigma \notin \{\Sigma_{1,1}, \Sigma_{1,2}, \Sigma_{2,0}, \Sigma_{0,4}\}$ .

**Theorem 12.2.** (Luo) Any automorphism of  $C(\Sigma_{1,2})$  preserving the set of vertices represented by separating loops is induced by a self-homeomorphism of the surface  $\Sigma_{1,2}$ .

We need the following result from [5].

**Theorem 12.3.** (Chas - Krongold) Let  $S$  be an oriented surface with non-empty boundary and let  $x$  be a free homotopy class of curves in  $S$ . Then  $x$  contains a simple representative if and only if  $\langle x, x^3 \rangle = 0$ , where  $x^3$  is the conjugacy class that wraps around  $x$  three times.

**Lemma 12.4.** Let  $x$  be a free homotopy class of oriented simple closed curves. Then  $x$  has a non-separating representative if and only if there exist simple class  $y$  such that the minimal intersection number of  $x$  and  $y$  is equal to one.

*Proof.* If  $x$  has a non-separating representative, the existence of  $y$  with minimal intersection number equal to one can be proved as in the proof of Lemma 5.1. Conversely, if  $x$  contains a separating representative, then the minimal intersection number of  $x$  and any other class is even. ■

**Theorem 12.5.** Let  $\Omega$  be a bijection on the set  $\pi^*$  of free homotopy classes of closed curves on an oriented surface. Suppose the following

- (1)  $\Omega$  preserves simple curves.
- (2) If  $\Omega$  is extended linearly to the free  $\mathbb{Z}$  module generated by  $\pi^*$  then  $\Omega$  preserves the Goldman Lie bracket. In symbols  $[\Omega(x), \Omega(y)] = \Omega([x, y])$  for all  $x, y \in \pi^*$ .
- (3) For all  $x \in \pi^*$ ,  $\Omega(\overline{x}) = \overline{\Omega(x)}$ .

Then the restriction of  $\Omega$  to the subset of simple closed curves is induced by an element of the mapping class group. Moreover, if  $\Sigma \notin \{\Sigma_{1,1}, \Sigma_{1,2}, \Sigma_{2,0}, \Sigma_{0,4}\}$  then the restriction of  $\Omega$  to the subset of simple closed curves is induced by a unique element of the mapping class group.

*Proof.* Since  $\Omega(\overline{x}) = \overline{\Omega(x)}$ ,  $\Omega$  induces a bijective map  $\widehat{\Omega}$  on  $\widehat{\pi} = \{x + \overline{x} : x \in \pi^*\}$ .

Since  $\Omega$  preserves the oriented Goldman bracket and the "change of direction", then it preserves the unoriented Goldman bracket. Let  $x$  be a class of oriented curves that contains a simple representative, let  $y$  be any class.

Since  $\widehat{\Omega}$  preserves the unoriented Goldman Lie bracket, by Theorem 8.2, the minimal intersection number of  $x$  and  $y$  equals the minimal intersection number of  $\widehat{\Omega}(x)$  and  $\widehat{\Omega}(y)$ . Thus, by Theorem 12.1, if the surface is not the torus with two holes,  $\widehat{\Omega}$  is induced by an element of the mapping class group (Observe that the "special" curve complexes are covered by this property). Moreover if the surface is not in  $\{\Sigma_{1,1}, \Sigma_{1,2}, \Sigma_{2,0}, \Sigma_{0,4}\}$  then  $\widehat{\Omega}$  is induced by unique element of the mapping class group.

Now we study the case of the torus with two holes. By Lemma 12.4,  $x$  is a separating simple closed curve, if and only if  $\Omega(x)$  is separating. Thus,  $\widehat{\Omega}$  maps bijectively the set of unoriented separating simple closed curves onto itself. Hence, by Theorem 12.2,  $\widehat{\Omega}$  is induced by an element of the mapping class group. ■

By arguments similar to those of Proposition 12.5, one can prove the following.

**Theorem 12.6.** *Let  $\Gamma$  be a bijection on the set  $\widehat{\pi}$  of unoriented free homotopy classes of closed curves on an oriented surface. Suppose the following*

- (1)  $\Gamma$  preserves simple curves.
- (2) If  $\Gamma$  is extended linearly to the free  $\mathbb{Z}$  module generated by  $\pi^*$  then  $\Gamma$  preserves the unoriented Goldman Lie bracket. In symbols  $[\Gamma(x), \Gamma(y)] = \Gamma([x, y])$  for all  $x, y \in \widehat{\pi}$ .

*Then the restriction of  $\Gamma$  to the subset of simple closed curves is induced by an element of the mapping class group. Moreover, if  $\Sigma \notin \{\Sigma_{1,1}, \Sigma_{1,2}, \Sigma_{2,0}, \Sigma_{0,4}\}$  then the restriction of  $\Gamma$  is induced by a unique element of the mapping class group.*

**Theorem 12.7.** *Let  $\Omega$  be a bijection on the set  $\pi^*$  of free homotopy classes of curves on an oriented surface with non-empty boundary. Suppose the following*

- (1) If  $\Omega$  is extended linearly to the free  $\mathbb{Z}$  module generated by  $\pi^*$  then  $[\Omega(x), \Omega(y)] = \Omega([x, y])$  for all  $x, y \in \pi^*$ .
- (2) For all  $x$  in  $\pi^*$ ,  $\Omega(\overline{x}) = \overline{\Omega(x)}$ .
- (3) For all  $x$  in  $\pi^*$ ,  $\Omega(x^3) = \Omega(x)^3$ .

*Then the restriction of  $\Omega$  to the set of free homotopy classes with simple representatives is induced by an element of the mapping class group. Moreover, if  $\Sigma \notin \{\Sigma_{1,1}, \Sigma_{2,0}, \Sigma_{0,4}\}$  then  $\Omega$  is induced by a unique element of the mapping class group.*

*Proof.* Let  $x$  be an oriented closed curve. By hypothesis,  $[\Omega(x), \Omega(x)^3] = \Omega([x, x^3])$ . Thus,  $[x, x^3] = 0$  if and only if  $[\Omega(x), \Omega(x)^3] = 0$ . Thus by Theorem 12.3,  $\Omega(x)$  is simple if and only if  $x$  is simple. Then the result follows from Theorem 12.5. ■

All these results "support" Ivanov's statement in [18]:

**Metaconjecture** *”Every object naturally associated to a surface  $S$  and having a sufficiently rich structure has  $\text{Mod}(S)$  as its group of automorphisms. Moreover, this can be proved by a reduction theorem about the automorphisms of  $C(S)$ .”*

In this sense, the Goldman Lie bracket combined with the power maps, have a ”sufficiently rich” structure.

### 13 Questions and open problems

**Problem 13.1.** Etingof [11] proved using algebraic tools that the center of Goldman Lie algebra of a closed oriented surface consist in the one dimensional subspace generated by the trivial loop. On the other hand, if the surface has non-empty boundary, it is not hard to see that linear combinations of conjugacy classes of curves parallel to the boundary components are in the center. Hence, it seems reasonable to conjecture that the center consist on linear combinations of conjugacy classes of boundary components. It will be interesting to use the results of this work to give a complete characterization of the center of the Goldman Lie algebra.

If  $u$  is an element of the vector space of unoriented curves on a surface, and  $u$  is a linear combination of classes that admit a simple representative then  $u$  is not in the center of the Goldman Lie algebra. To prove this, we need to combine our results with the fact that given two different conjugacy classes  $a$  and  $b$  that admit a simple representative, there exists a simple conjugacy class  $c$  such that the intersection numbers  $i(a, c)$  and  $i(b, c)$  are distinct (see [12] for a proof that such  $c$  exists). Nevertheless, by results of Leininger [21] we know that this argument cannot be extended to the cases of elements of the base that only have self-intersecting representatives.

□

**Problem 13.2.** As mentioned in the Introduction, Abbaspour [1] studied whether a three manifold is hyperbolic by means of the generalized Goldman Lie algebra operations. He used free products with amalgamations for this study. We wonder if it would be possible to combine his methods with ours, to give a combinatorial description of the generalized String Topology operations on three manifolds. In this direction, one could study the relation between number of connected components of the output of the Lie algebra operations and intersection numbers.

□

**Problem 13.3.** We showed that subspaces  $W(w_1, w_2, \dots, w_n)$  defined in Subsection 11.1 are  $\text{ad}_x$ -invariant. Let  $z$  be a representative of a conjugacy class in  $W(w_1, w_2, \dots, w_n)$ . It would be interesting to define precisely and study the ”number of twists around  $x$ ” of the sequence  $(\text{ad}_x^n(z))_{n \in \mathbb{Z}}$  and how this number of twists

changes under the action of  $\text{ad}_x$ . Observe also that the Dehn twist around  $x$  increases or decreases the number of twists at a faster rate. These problems are related to the discrete analog of Kerckhoff's convexity [19] found by Luo [24].  $\square$

**Problem 13.4.** The Goldman Lie bracket of two conjugacy classes, one of them simple, has no cancellation. On the other hand, there are examples (for instance Example 9.1) of pairs of classes with bracket zero and non-zero minimal intersection number. We wonder which is the topological characterization of pairs of intersection points of two curves, for which the corresponding term of the bracket cancel. In other words, what "causes" cancellation. The tools to answer to this question may involve the study Thurston's compactification of Teichmüller space in the context of Bonahon's work on geodesic currents [2].  $\square$

**Problem 13.5.** Dylan Thurston [28] proved a suggestive result: Let  $m$  be a union of conjugacy classes of curves on an orientable surface and let  $s$  be the conjugacy class of a simple closed curve. Consider representatives  $M$  of  $m$ ,  $S$  of  $s$  which intersect (and self-intersect) in the minimum number of points. Denote by  $P$  one of the self-intersection points of  $M$ . Denote by  $M_1$  and  $M_2$  the two possible ways of smoothing the intersection at  $P$ . Denote by  $m_1$  and  $m_2$  respectively the conjugacy classes of  $M_1$  and  $M_2$ . Then Dylan Thurston's result is

$$i(m, s) = \max(i(m_1, s), i(m_2, s)).$$

These two "smoothings" are the two local operations one makes at each intersection point to find a term of the unoriented Goldman Lie bracket (when the intersection point  $P$  is not a self-intersection point of a curve). It might be interesting to explore the connections of Dylan Thurston's result and our work.  $\square$

**Remark 13.6.** In a subsequent work we will give a combinatorial description of the set of cyclic sequences of double cosets under the action of  $\text{ad}$ . Also, we will study under which assumptions the cyclic sequences of double cosets are a complete invariant of a conjugacy class.  $\square$

## References

- [1] H. Abbaspour, *On string topology of 3-manifolds*, Topology, **44** no 5 , (2005), 1059–1091. [arXiv:0310112](#)
- [2] F. Bonahon, *The geometry of Teichmüller space via geodesic currents*, Invent. Math. **92** (1988), 139-162.

- [3] M. Chas, *Combinatorial Lie bialgebras of curves on surfaces*, Topology **43**, 543-568,(2004), [arXiv: 0105178v2 \[math.GT\]](#)
- [4] M. Chas, *Minimal intersection of curves on surfaces*, [arXiv: 0706.2439v1 \[math.GT\]](#)
- [5] M. Chas and F. Krongold, *An algebraic characterization of simple closed curves*, [arXiv:0801.3944v1 \[math.GT\]](#)
- [6] M. Chas and D. Sullivan, *String Topology*, [arXiv: 9911159 \[math.GT\]](#) Annals of Mathematics (to appear).
- [7] M. Chas and D. Sullivan, *Closed operators in topology leading to Lie bialgebras and higher string algebra*, The legacy of Niels Henrik Abel, (2004) 771-784 [arXiv: 0212358 \[math.GT\]](#)
- [8] E. C. Cohen, *Combinatorial group theory, a topological approach*, London Mathematical Society Student Texts, **14**, Cambridge University Press, 1989.
- [9] M. Cohen and M. Lustig, *Paths of geodesics and geometric intersection numbers I*, Combinatorial Group Theory and Topology, Utah, 1984, Ann. of Math. Studies **111**, Princeton Univ. Press, Princeton, (1987), 479-500.
- [10] W. Dicks and M. J. Dunwoody, *Groups acting on graphs*, Cambridge Stud. Adv. Math. **17**, CUP, Cambridge, 1989.  
Errata at <http://mat.uab.cat/~dicks/DDerr.html>
- [11] P. Etingof, *Casimirs of the Goldman Lie algebra of a closed surface*, Int. Math. Res. Not. (2006).
- [12] A. Fathi, F. Laudenbach A and V. Poenaru, *Travaux de Thurston sur les surfaces*, Astérisque, **66-67**, Soc. Math. France, Paris,(1979)
- [13] W. Goldman, *Invariant functions on Lie groups and Hamiltonian flows of surface group representations*, Invent. Math. **85** (1986), 263-302.
- [14] J. Harer and R. Penner, *Combinatorics of train tracks*, Ann. Math. Studies **125**, Princeton University Press, Princeton, NJ, (1992)
- [15] W. J. Harvey, *Geometric structure of surface mapping class groups*, Homological group theory (Proc. Sympos., Durham, 1977), 255–269. Cambridge Univ. Press, Cambridge, 1979.
- [16] A. Hatcher, *Algebraic topology*, Cambridge University Press, 2002.



- [17] N. Ivanov, *Automorphisms of complexes of curves and of Teichmüller spaces*, Progress in knot theory and related topics, 113–120. Hermann, Paris, 1997.
- [18] N. Ivanov, *Fifteen problems about the Mapping Class group*, Problems on mapping class groups and related topics, (editor Benson Farb), 71-80, Providence, R.I., American Mathematical Society, (2006).
- [19] S. Kerckhoff, *The Nielsen realization problem*, Ann. Math. **117**,(1983) 235-265.
- [20] M. Korkmaz, *Automorphisms of complexes of curves on punctured spheres and on punctured tori*. Topology Appl., **95**(2):85–111, (1999).
- [21] C. Leininger, *Equivalent curves in surfaces*, Geometriae Dedicata, **102**, 1, 2003, 151-177(27) [arXiv:math.GT/0302280v1](#) [[math.GT](#)]
- [22] F. Luo, *On non-separating simple closed curves in a compact surface*, Topology **36**, No 2, pp 381-410, (1997).
- [23] F. Luo, *Automorphisms of the complex of curves*. Topology **39**(2), 283–298, (2000). [arXiv: 9904020](#) [[math.GT](#)]
- [24] F. Luo, *Some Applications of a Multiplicative Structure on Simple Loops in Surfaces*, Knots, Braids and Mapping Class-Groups-Papers Dedicated to Joan S. Birman, (editors Gilman, Menasco, and Lin), 85-93, AMS/IP Studies in Advanced Mathematics. (2001). [arXiv: 9907059](#)
- [25] R.C. Lyndon and R.E. Schupp, *Combinatorial Group Theory*, Springer-Verlag, Berlin Heidelberg New York, 2001.
- [26] W. Magnus, W. Karrass and D. Solitar, *Combinatorial Group Theory, presentations of groups in terms of generators and relations*, Dover, 1966.
- [27] D. Margalit, *Automorphisms of the pants complex*, Duke Mathematical Journal **121**(3) 457-479, 2004. [arXiv: 0201319](#) [[math.GT](#)]
- [28] D. Thurston, *On geometric intersection of curves in surfaces*, preprint, <http://www.math.columbia.edu/~dpt/writing.html>
- [29] W. Thurston, *On the geometry and dynamics of diffeomorphisms of surfaces*, Bull. Amer. Math. Society **19**, (1988) 417-438, <http://projecteuclid.org/euclid.bams/1183554722>

MOIRA CHAS, DEPARTMENT OF MATHEMATICS, SUNY AT STONY BROOK,  
STONY BROOK, NY, 11794

*E-mail address:* [moira@math.sunysb.edu](mailto:moira@math.sunysb.edu)